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The Micro-Macro Link for the Theory of Structural Balance

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I consider the principle of structural balance *that is commonly characterized with the aphorisms: "The friend of a friend is a friend, the friend of an enemy is an enemy, the enemy of a friend is an enemy, and the enemy of an enemy is a friend." I study what patterns of friendship and hostility emerge at the macro-level when actors at the micro-level make friends and enemies in accordance with this principle. Recent studies have drawn attention to configurations that are* imbalanced *with many triadic relations violating structural balance yet* jammed *because no change in sentiment in any one relation can accomplish a net reduction in the number of violations of structural balance. The existence of such jammed states suggests that individual behavior consistent with structural balance need not aggregate to system-wide satisfaction of the principle. To investigate this I employ a best-response model of sentiment change on a fixed social network. I show that under a broad set of model conditions only configurations in which all triadic relations satisfy the structural balance principle can emerge. In a close-knit community in which all actors maintain relations with all other actors such a balanced configuration must consist of either one friendship clique or multiple antagonistic groups.*

Keywords: balance theory, dynamic models, social networks

1. INTRODUCTION

"The friend of a friend is a friend, the friend of an enemy is a friend, the enemy of a friend is an enemy, and the enemy of an enemy is a

I thank the editors Vincent Buskens, Werner Raub and Marcel van Assen for extraordinarily helpful comments on earlier drafts and for their dedication to this special issue. I also wish to extend my gratitude to Ko Kuwabara, Patricio Gallardo, Tadeusz Sozański and two anonymous reviewers for very useful suggestions.

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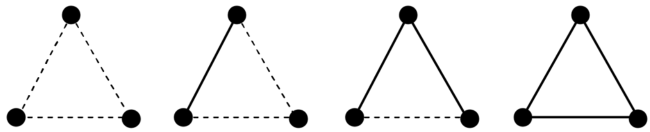

FIGURE 1 Four signed triads. Solid lines represent positive edges, dashed lines are negative. Under structural balance, triads (a) and (c) are imbalanced, while (b) and (d) are balanced. Under generalized balance, triad (c) is imbalanced, while triads (a), (b) and (d) are balanced.

friend." These aphorisms provide a simple heuristic for how parties may swear allegiance or declare war based on outstanding feuds with and loyalties to third parties. For example, scholars refer to behavior following this principle by hooligans and urban youth as the *Bedouin syndrome* (Harrison, 1974; Robins and Cohen, 1978; Dunning, Murphy, and Williams, 1986; Roversi, 1991; Cohen, 1997; Russell, 2004; Maniglio, 2007). Thus forming spontaneous ad hoc alliances, hooligans and urban youth can put together large armies in the absence of formal organization (Robins and Cohen, 1978; Dunning, 1999). The general question of whether the emergence of team structure at the macro-level is a necessary outcome of such micro-level behavior is the focus of the present study. For long this question was considered answered with Cartwright and Harary's (1956) "structure theorem" (p. 286) which suggested that any conflict must involve two opposed factions, until a recent discovery by Antal, Krapivsky, and Redner (2005).

Namely, the heuristic can be expressed as a condition for a signed graph (Rapoport, 1963), in which case scholars speak of *structural balance* (Harary, 1954; Cartwright and Harary, 1956) in reference to Heider's balance theory (Heider, 1946).[1] If any two actors are either friendly or hostile toward one another, then a fully connected triad can take on any of four sentiment patterns. Figure 1 shows the four corresponding signed triads, where a solid line represents a positive edge and a friendship and a dashed line represent a negative edge and thus stand for enmity. We will refer to these triads in the remainder of the article as respectively $---$, $+--$, $++-$, and $+++$. Triads $+--$ and $+++$ are balanced, while triads $---$ and $++-$ are imbalanced.

[1]Heider (1946) defined balance in a more general scenario in which triads can contain both individuals as well as objects and relationships are directed. A separate line of scholarship employs this original notion of balance (e.g., Hummon and Doreian, 2003; Montgomery, 2009).

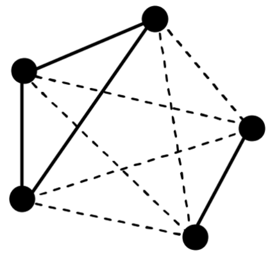

FIGURE 2 A complete signed graph satisfying structural and generalized balance.

Cartwright and Harary's result (Harary, 1954, pp. 143–144; Cartwright and Harary, 1956, p. 286)[2] speaks to the question of what sentiment configurations can be expected at the macro-level under the rule of structural balance. The result relates the absence of imbalanced triads to *clusterability* (Wasserman and Faust, 1994, p. 234) of a complete signed graph. A complete signed graph is a signed graph in which all possible ties are present. A signed graph is clusterable if one can partition its nodes into a finite number of subsets such that positive edges connect only nodes in the same subset and negative edges connect only nodes from distinct subsets. Cartwright and Harary's (1956) result is that in any complete signed graph all triads are balanced if and only if the graph is clusterable into one or two clusters. That is, either all ties are positive or a bipartition exists with only positive within-group ties and only negative between-group ties (Fig. 2). Note that a complete signed graph that can be partitioned into more than two clusters cannot be balanced as any three nodes from any three distinct clusters form an imbalanced $---$ triad. The result suggests that under the rule of structural balance, patterns of hostility must involve conflict situations with two opposed factions. Such consolidated affiliations have long been argued to destabilize a social system: once group lines are clearly drawn the stage is set for overt conflict (Simmel, 1910, 1955; Coser, 1956; Dahl, 1956; Dahrendorf, 1959; Lipset, 1959; Lijphardt, 1968; Blau and Schwartz, 1984; Flap, 1997).

Two recent studies draw attention to a fascinating observation (Antal, Krapivsky, and Redner, 2005; Marvel, Strogatz, and Kleinberg, 2009). There exist configurations that, unlike Figure 2, contain imbalanced triads, yet, like Figure 2, no single sign change can increase net

[2]Cartwright and Harary (1956) also prove that a signed graph, which may be incomplete, is clusterable into one or two groups if and only if every cycle in the graph (not just every triad) contains an even number of negative edges.

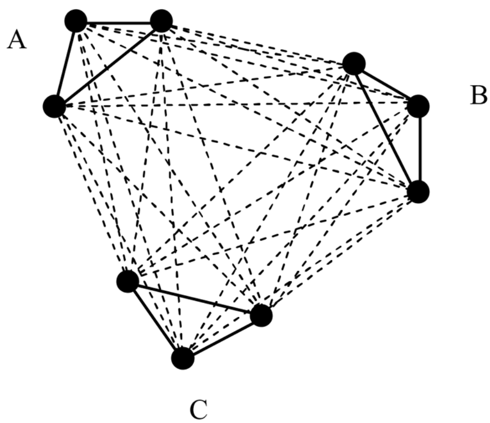

FIGURE 3 A complete signed graph satisfying generalized balance but not structural balance. No sign change decreases the number of violations of structural balance.

balance. An example of what they refer to as a "jammed state" is shown in Figure 3. Any sign change would strictly reduce net balance. Namely, any change from $-$ to $+$ makes three triads balanced at the expense of introducing four new imbalanced triads, while any change from $+$ to $-$ would make seven triads imbalanced while making none balanced. For example, if one tie between an actor in group A and an actor in group B were turned from negative to positive, then this would reduce the number of negative ties in any triad with a third actor in group C from three down to two, rendering it balanced. There are three such triads. In the remaining four triads that involve the focal tie the third actor is a member of group A or B and in any such triad the number of negative ties is brought down from two to one, thereby rendering it imbalanced. All positive ties in the Figure 3 graph only involve balanced triads. Changing the sign of any such positive tie would thereby render all seven triads it is involved in imbalanced. Thus neither a negative tie turned positive nor a positive tie turned negative increases net balance. Antal et al. (2005, pp. 6–7) characterize a general class of jammed states that can be partitioned into three groups of roughly equal size, to which Figure 3 belongs. An example from another general class of jammed states identified by Marvel et al. (2009, pp. 2–3) is shown in Figure 4. Each edge is involved in exactly two balanced and two imbalanced triads so that changing its sign renders the former imbalanced while rendering the latter balanced, leaving overall balance unaffected. In contrast to Figures 2 and 3, Figure 4 is not clusterable. Such jammed states can be taken to

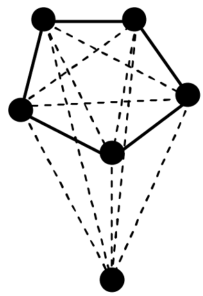

FIGURE 4 A complete signed graph satisfying neither structural nor generalized balance. No sign change decreases the number of violations of structural balance.

represent situations in which divided loyalties obscure group boundaries, making organized conflict difficult. The existence of such jammed states suggests that individual behavior consistent with structural balance may nevertheless fail to aggregate to system-wide satisfaction of structural balance.

The landscape of loyalty patterns is thus rife with configurations that qualitatively vary in both their degree of structural balance and their theoretical potential for group conflict. The task at hand, to identify those system-level patterns that emerge when individuals act consistently with structural balance, requires explicit behavioral rules for how actors alter sentiment in the relations adjacent to them. Implicit in the concept of a jammed state is a model of tie-by-tie change. That is, actors sequentially modify the sign of one of their ties, befriending a former enemy or upsetting an existing friendship, such that net balance is increased. The process comes to a natural end when no sign change can raise net balance. If once halted any imbalances remain, jamming must have occurred.

This actor model that is implicit in the concept of a jammed state employs a weak notion of stability. Here a comparison with the literature on dynamic network models is informative, which rather than signed graphs in which actors change *sentiment* considers unsigned graphs in which actors add and delete *ties*. The weakest stability notion in this literature for models of tie change is *pairwise stability* (Jackson and Wolinsky, 1996), which closely matches the stability notion for sign change that is implicit in the Antal et al. (2005) and Marvel et al. (2009) studies. Namely, a network is pairwise stable if no actor can improve by removing an adjacent tie or by adding an adjacent tie to another actor who is not negatively affected by the creation of this tie. Arguments for two expansions of individual action space have been advanced in this literature. Studies typically find that each of the two expansions dramatically shrinks the number of stable configurations. First, actors may take actions that affect more than one adjacent tie at once as incorporated in the stronger stability concepts *strong pairwise stability* (Gilles and Sarangi, 2003) and *unilateral stability* (Buskens and van de Rijt, 2008). A child may turn on her best friend by joining a group of bullies, and war is regularly declared on multiple nations at once. For example, Figure 5, a jammed state, becomes "unjammed" when actors can change multiple ties at once. This is illustrated in Figure 6. Figure 6 shows what happens if a_1 befriends her three former enemies, thus mimicking c_1. This act reduces the number of imbalanced triads involving a_1 from nine down to six. When actors can change sentiment in multiple ties at once Figure 5 is no longer jammed.

A second reasonable behavioral assumption is that actors occasionally deviate and change an affiliation that does not decrease local imbalance. Such occasional deviations may represent bounded rationality or idiosyncracies in personal behavior (e.g., Young, 1998; Foster and Young, 1990). A single deviation can be enough to unjam the Figure 4 network. For example, suppose a negative tie inside the 5-cycle were turned into a positive tie. This has the effect that two other enemy-friend swaps inside the 5-cycle would now reduce imbalance, triggering a cascade that ends with a balanced network in which the only inimical relations are with the sixth actor.

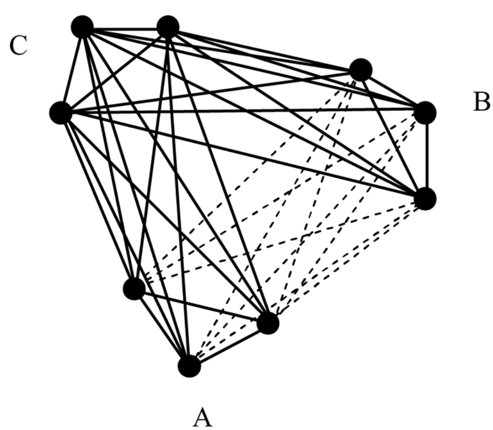

FIGURE 5 A complete signed graph satisfying neither structural nor generalized balance. No sign change decreases the number of violations of either structural or generalized balance.

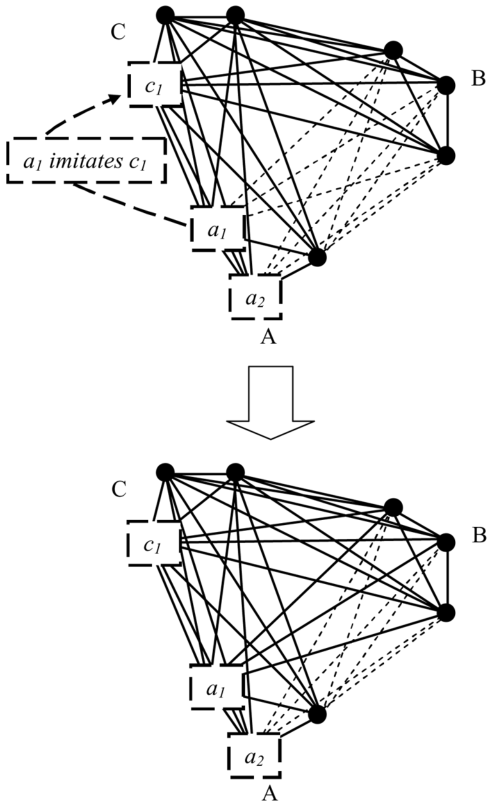

FIGURE 6 Example of an imbalance-reducing action with negative externalities in the complete graph from Figure 5. By imitating c_1, a_1 decreases the number of imbalanced triads involving herself but increases the number of imbalanced triads involving a_2.

In addition to investigating which sign configurations emerge under these two alternative actor assumptions, there are two other important generalizations we explore. First, we consider a modification of the balance principle proposed by Davis (1967) where the $---$ triad consisting of three negative relations is now also balanced. In contrast to structural balance, under a "generalized balance" regime the enemy of an enemy can then be either an enemy or a friend. Second, in some settings such as employees in an organization actors only interact with a fixed subset of others according to some formal organization structure. For such settings to fall within the scope of our analysis also incomplete networks must be studied.

2. LITERATURE

The extent to which sentiment and loyalty are organized around the structural balance principle has been investigated in a variety of contexts. These include international relations (Moore, 1978; Antal et al., 2005, 2006; Maoz, Terris, Kuperman, and Talmud, 2007; Brandes, Lerner, and Snijders, 2009), occurrences of avoidance and joking in Bushmen kinship relations (Hage, 1976), and patterns of liking and disliking in social groups such as among naval enlisted personnel (Kogan and Tagiuri, 1958), among veterans living in a domiciliary building (Davol, 1959), in fraternities (Doreian and Krackhardt, 2001), and in online social networks (Leskovec, Huttenlocher, and Kleinberg, 2010a, 2010b; Szell, Lambiotte, and Thurner, 2010). Evidence is equivocal. Taken as a whole it tends to find an overrepresentation of $+++$ and $+--$ triads and a strong underrepresentation of $++-$ triads, but often no underrepresentation of $---$ triads. These results are a better fit with the generalized version of the balance principle proposed by Davis (1967) which drops the requirement that the enemy of an enemy be a friend, thus permitting $---$ triads. In our analysis we will consider both structural and generalized balance.

As stated before, Cartwright and Harary (Harary, 1954, pp. 143–144; Cartwright and Harary, 1956, p. 286) showed that under a structural balance regime, complete graphs are balanced if and only if all edges are positive or it can be partitioned into *two* clusters (e.g., Fig. 2). Davis formulated a related result for complete graphs under a generalized balance regime (Davis, 1967, p. 181). Balance in all triads is equivalent to the graph containing only positive edges or being clusterable into any number of clusters (e.g., Fig. 3).

A growing body of literature on models of network evolution has identified a host of scenarios in which local utility maximization through network modifications by agents does not produce global maximization (see Goyal, 2006; Jackson, 2008, ch. 6, for overviews). For example, Jackson and Wolinsky (1996) find that in the absence of central organization agents may build inefficient communication networks. Buskens and van de Rijt (2008) show that agents who attempt to broker other agents by placing themselves in "structural holes" (Burt, 1992) may end up in unnecessarily dense networks containing many redundant ties. Doğan, van Assen, van de Rijt, and Buskens (2009) demonstrate how the practice of maintaining rival connections with alternative exchange parties as a negotiation tactic may lead to collective overspending on relationship maintenance. An important difference with the current study is that we do not investigate the evolution of network structure. Rather, we study the evolution of sign changes while leaving the network fixed.

Apart from the two studies mentioned before (Antal et al., 2005; Marvel et al., 2009), several other studies have investigated dynamic models that follow balance-theoretic principles (Abell and Ludwig, 2007; Gawroński, Gronek, and Kulakowski, 2005; Hummon and Doreian, 2003; Kulakowski, Gawroński, and Gronek, 2005; Kuwabara, 2006; Ludwig and Abell, 2009; Montgomery, 2009; Wang and Thorngate, 2003). Many of the models in these studies prevent jamming from occurring by assuming that (i) actors are given an exit option, allowing them to eliminate ties or neutralize sentiment, or by assuming that (ii) actors can have ambivalent evaluations of others. There are important reasons to study the case where (i) and (ii) do not apply. A variety of contexts, including many of the aforementioned empirically investigated contexts, place institutional or spatial constraints on actors so that they cannot avoid frequently running into the same alters and having to state their evaluations of third parties in unambiguous terms. They repeatingly encounter these same actors and may find themselves in situations where they have to choose sides. These include children in a classroom, employees in an organization, members of a sorority, gangs in a neighborhood, bands of hooligans, and sovereign states. Of sociological interest is how balancing plays out in these scenarios.

3. A BEST-RESPONSE MODEL

Following Antal et al. (2005) and Marvel et al. (2009), the balancing process is modeled as sequential sign change on a fixed undirected network. We specify an actor utility function that increases in the number of balanced triads that involve the actor. The two results we present pertain to both structural and generalized balance regimes. The assumptions that networks are complete and that actors never deviate are relaxed for the second result.

Consider a finite set of actors $N = \{1, 2, \ldots, n\}$. Actors cannot have ties with themselves (ties are nonreflexive) and if i has a relation with j, then j also has a relation with i (ties are undirected). The set of all possible relations between the actors is then the set X^N of all two-element subsets ij of N. A signed graph (hereafter just "graph") $G = (N, X, F)$ combines a set of actors N with a set of relations $X \subseteq X^N$, a subset $F \subseteq X$ of which are positive, relations in its complement $E = X \backslash F$ being negative. Denote by X_i the subset of all relations in X involving actor i and by F_i the subset of all positive relations in F

involving i. Two actors i and j are *sign-equivalent* on G if $ik \in F_i \Leftrightarrow jk \in F_j$ for all $ik, jk \in X$.

Under a *structural balance regime*, any closed triad $ijk := \{ij, ik, jk\} \subseteq X$ is *balanced* if $|F \cap ijk| \in \{1, 3\}$, and imbalanced otherwise. Alternatively, under a *generalized balance regime*, any triad $ijk := \{ij, ik, jk\} \subseteq X$ is balanced if $|F \cap ijk| \in \{0, 1, 3\}$, and imbalanced otherwise. For both regimes, denote by $t(G, i)$ the number of imbalanced triads in G that involve i and by $t(G, ij)$ the number of imbalanced triads in G that involve i and j. Actor i's utility $u_i(G)$ strictly decreases in $t(G, i)$. A graph G is balanced if $\sum_i t(G, i) = 0$.[3]

An actor can only change signs in relations involving herself. We denote by $m > 0$ the maximum number of signs an actor can change at any one time in her ego-network. Any $S_i \subseteq X_i$ for which $|(F_i \backslash S_i) \cup (S_i \backslash F_i)| \leq m$ is then a *strategy* on graph $G = (N, X, F)$ for actor i. Strategy S_i^* is a *best response* to F if $S_i^* \in \text{argmax}_{\{Si\}}\ u_i(N, X, (F \backslash F_i) \cup S_i)$. A graph G is *stable* if F_i is a best response to F for all i. The "jammed states" in Antal et al. (2005) and Marvel et al. (2009) are stable graphs under special conditions $m = 1$ and $X = X^N$. We consider best response dynamics, in which actors sequentially make sign changes to the graph that are best responses. An *improving path* is a finite sequence of Z graphs $\{G^z = (N, X, F^z)\}$ and a sequence of Z–1 actors $\{i^z \in N\}$ such that $(F^z \backslash F^{z+1}) \cup (F^{z+1} \backslash F^z)$ is a strategy on G^z for actor i^z and $u_{i^z}(G^{z+1}) > u_{i^z}(G^z)$ for any $z < Z$. No $G^z \neq G^Z$ in an improving path can be stable. We also informally define stochastic stability (Foster and Young, 1990). A graph G is a *stochastically stable state* if G maintains a nonzero probability of occurrence in smoothed best response dynamics under vanishing noise. As opposed to best response dynamics, in smoothed best response dynamics, bad responses do occur but better responses are more likely. As noise vanishes, non-best responses become infinitesimally rare and stochastically stable states are the only states we could observe in the long run.

4. RESULTS

Antal et al. (2005) and Marvel et al. (2009) show that for complete graphs under the structural balance regime, the number of jammed states outnumbers the number of balanced states. Almost all stable states are imbalanced for large networks. The Figure 3 and 4 networks are only jammed under a structural balance regime but not under a

[3]Cartwright and Harary (1956) consider a graph balanced if all cycles are positive, not just all triads. For complete graphs balance of all triads implies balance of all cycles but for incomplete graphs it does not.

generalized balance regime. Figure 5 shows that jammed states also exist under a generalized balance regime. The example in Figure 5 consists of two antagonistic groups, A and B, both allied with a third group, C. All imbalanced triads in Figure 5 contain two positive edges and one negative edge. A sign change of any positive between-group tie or any negative tie involving any actor makes three previously imbalanced triads involving that tie balanced, but the four other triads involving that tie become imbalanced. Any positive within-group tie turned negative introduces imbalance into all seven triads involving that tie. It is straightforward to construct states such as those in Figure 5 for larger network sizes. Any configuration of three roughly equally sized groups with positive within-group edges, positive between-group edges for two of the three pairs of groups, and negative edges between the third pair is jammed under a generalized balance regime. Similar configurations can be construed for more than three groups. Note that Figure 5 is also jammed under structural balance.

Under the weakest stability notion there thus exists an abundance of jammed states, both when considering structural balance as well as generalized balance. Theorem 1 claims that when we strengthen the stability notion by allowing multiple simultaneous sign changes ($m = n - 1$), all jammed states break down. Complete networks are then necessarily balanced. A proof is given in the Appendix.[4]

Theorem 1. *Under both a structural balance regime as well as a generalized balance regime, for* $m = n - 1$ *and* $X = X^N$*, only balanced graphs are stable, and for any imbalanced graph* $G^1 = (N, X^N, F^1)$ *there exists an improving path* $(\{G^Z\}, \{i^{Z-1}\})$ *such that* G^Z *is balanced.*

The proof of theorem 1 proceeds by showing that in any imbalanced complete graph some actor can reduce local imbalance by making herself sign-equivalent to some other actor. Starting from any imbalanced complete graph there must then exist an improving path to a balanced graph. Theorem 1 suggests that if actors attempt to reduce imbalance in their ego-networks by changing the signs of one or more of their relations then they will necessarily succeed. The balancing process cannot jam before global balance is reached.

The proof of theorem 1 in the Appendix suggests a balance-improving option that is always available in an imbalanced complete network. Namely, there is always some actor who can improve balance

[4]Although the theorem considers $m = n - 1$, many of the configurations that Antal et al. (2005) and Marvel et al. (2009) identify that are jammed for $m = 1$ already unjam for intermediate values of m.

by "copying" a more balanced friendship configuration of some other actor. Figure 6 illustrates this option for the Figure 5 network, which is imbalanced under both balance regimes. Actor a_1 imitates actor c_1 and changes sentiment in relations with third parties accordingly. As a consequence, the number of imbalanced triads involving actor a_1 is reduced by three.

Theorem 1 demonstrates how the stability of jammed configurations breaks down in complete networks when actors are allowed to change the signs of more than one relation at a time. Although the theorem is limited to the scenario where actors can make as many simultaneous sign changes as they want ($m = n - 1$), many of the configurations that Antal et al. (2005) and Marvel et al. (2009) identify that are jammed when actors can make only one sign change at a time ($m = 1$) already become unjammed when there is an intermediate limit on the number of simultaneous tie changes ($1 < m < n - 1$).

We now investigate the effects of an alternative strengthening of the stability notion, namely by considering smoothed best response dynamics in which actors occasionally deviate. Like theorem 1, theorem 2 guarantees drift toward global balance whenever agents attempt to reduce imbalances in their ego-networks. We find that for either balance regime and for any network structure, complete or incomplete, and for any allowed number of simultaneous sign changes, only balanced networks can be stochastically stable. A proof can be found in the appendix.

Theorem 2. *Under both a structural balance regime as well as a generalized balance regime, given any actor set N with cardinality n and given any edge set $X \subseteq X^N$, for any maximum number of simultaneous sign changes $m > 0$, only balanced graphs $G = (N, X, F)$ can be stochastically stable.*

Actors' incentives in the games we consider are *not* aligned. They involve negative externalities, where neighbors see their local imbalance increase as a result of the focal actor's imbalance-reducing action. For example, in Figure 6, actor a_1's action has negative externalities for actor a_2 who sees the number of imbalanced triads he is involved in go up nine to 12. Nevertheless, any change in balance that individuals achieve by modifying their relations is exactly equal to the resulting global change in balance. This is evident from the observation that actors do not affect the balance of triads other than the ones they control. The consequence is that individual balance maximization while causing local negative externalities continually pushes the dynamic further in the direction of greater global balance.

4.1. The Issue of Consent

The occasional deviations from structural balance that occur under smoothed best responses prevent the process from getting stuck in jammed configurations. The presence of negative externalities, however, raises an issue of consent. In the best-response model developed here, agents force peace and war upon others. Yet, arguably, peace is conditional upon consent. This does not pose a problem for the scenario where actors make one tie change at a time ($m = 1$), as for any single tie change incentives are always aligned with the respective other actor. Namely, any change in triadic balance experienced by the focal actor as a result of that change is necessarily also experienced by the other actor. Consent by the other actor involved in the tie change can be expected. However, when actors consider multiple simultaneous tie changes ($m = n - 1$) negative externalities may exist.

Various types of consent giving can be distinguished that differ in terms of what new friends think will happen were they to decline the new friendship proposal. Consider first the scenario where they think that all other changes the focal actor is proposing to make will go through. In this case they will give consent only if in the new network including the new friendship with the focal actor they experience no less balance than in the new network excluding the new friendship. It can be demonstrated that in this scenario in an imbalanced complete graph there is always some actor who can propose a sentiment reconfiguration that can count on consent by all new friends. From theorem 1 we already know that there exists a proposal by some focal actor that strictly improves net balance to that actor. Imagine that one of this actor's new friends prefers the new network without the new friendship to the new network with the new friendship included. Since all triads involving the tie between the focal actor and the new friend also affect the focal actor, it follows that net balance for the focal actor must also be higher in the absence of the new friendship. So the alternative proposal without this new friendship improves net balance to the focal actor even more than the original proposal. Clearly, we can keep modifying the proposal until all new friends give consent, while in the process of doing so further improving the proposal to the focal actor. Hence, a balance-improving proposal that can count on consent always exists.

Alternatively, new friends may think that a veto would stop the entire proposal. To explore the effect of consent under this alternative scenario, I implemented the requirement of permission of a new friendship by that new friend in the following simulation. Each simulation run started in some random initial complete signed graph of small size (up to 25 actors). A random agent was then given the opportunity to reconfigure the pattern of signs in her relations but only in such a way that any new friend did not experience an increase in the number of imbalanced triads involving her.[5] The agent made that change that maximized local balance under this additional constraint. Then a next agent was chosen at random, and so on, until the network was balanced. All simulation runs converged to a balanced graph.

5. DISCUSSION

Through the lens of structural balance theory, group conflict can be viewed as emerging from the local decisions of actors on whom to befriend and whom to fight based on outstanding loyalties. In many contexts a group of actors must regularly meet one another: children in a classroom, employees in an organization, members of a sorority, gangs in a neighborhood, bands of hooligans, and sovereign states. Institutional and spatial constraints prevent the actors from dealing with conflicting loyalties by avoiding interaction with the enemies of friends or through ambiguation about where their loyalties lie. To reduce stress in their relations they may have no choice but to upset former friendships and make peace with former enemies. In these contexts, relations form a fixed network but the sentiment in these relations can change. I have investigated the micro-macro link for structural balance theory, asking what patterns of sentiment emerge when actors upset friendships and befriend enemies in accordance with the principle of structural balance.

Recent studies show that "jammed states" exist in which the pattern of positive and negative edges contains imbalanced triads yet no actor can change sentiment in any one tie without introducing imbalance in another triad. I have demonstrated how such jammed states also exist under what is referred to as "generalized balance," where the requirement that the enemy of an enemy be a friend is dropped. This suggests that a process in which actors repeatedly change relationship sentiment on a fixed network in accordance with structural balance or generalized balance may nevertheless get stuck in imbalanced configurations. To investigate this I have proposed a

[5]This form of consent where permission for a *sentiment* change is required by all new *friends* is analogous to the form of consent employed in the unilateral stability concept proposed by Buskens and van de Rijt (2008) where permission for a *network* change is required by all new *contacts*.

best-response model of sign changes on a fixed network. In this model actors seek to minimize the number of imbalanced triads they are involved in. I found that jamming only occurs under the weakest notion of stability. If actors consider actions that involve sentiment changes in multiple ties at once, balancing processes on a complete network necessarily lead to balance in all relations. Even if actors consider only one sentiment change at a time but occasionally deviate from the balance principle, again only balanced networks emerge in the long run. If all relations in these networks are present then they can be characterized as either a single friendship clique or multiple antagonistic friendship cliques. These results suggest that as relations improve between those with similar sentiment toward third parties and deteriorate between those who disagree on who they like, actors self-organize along party lines.

The present investigation into the macro-level consequences of balancing at the micro-level was performed under a number of different model conditions. These include structural as well as generalized balance, complete and incomplete networks, one or multiple simultaneous tie changes per actor, and no deviations versus occasional deviations from the balance principle. Two modifications of the model are worth exploring. First, actors who in real life regularly change loyalty incur the cost of developing a bad reputation. We may accordingly introduce a cost to sentiment change into the model. This raises the theoretical question of how high costs must be to jam balancing processes. Empirically, one could assess if changes in relationship sentiment are more likely if they accomplish not just a net reduction in imbalance but also some reduction of sufficient magnitude. Such an empirically suggested threshold value would be indicative of the presence of a cost to loyalty change.

A second worthwhile modification of the model is the introduction of a preference for friendship over inimicality. For this alternative model a question of interest becomes under what conditions group conflict can be resolved. Namely, one can imagine an initially balanced situation with two antagonistic cliques and a process of conflict resolution where some first movers with a strong preference for friendship become friends at the expense of introducing imbalance into the network. This paves the way for actors with a slightly weaker preference for friendship over enmity to follow their example, and so on until a single friendship clique is formed. Discontinuities in the distribution of strength of the preference for friendship, as in a polarized population consisting of doves and hawks, may then jeopardize a cascade of reconciliation.

A promising new venue for empirical testing of balance-theoretic predictions is the wealth of fine-grained temporal data from digital sources that have become available. Several recent studies have evaluated structural balance in the context of four empirical settings: relationships among contributors to an online encyclopedia, a product review website, a news website, and players in a large multiplayer online game (Leskovec et al., 2010a; Leskovec et al., 2010b; Szell et al., 2010). Leskovec et al. (2010a) investigate the prediction of a single friendship clique or bipartition into two antagonistic clusters in the first three contexts and find no support. There are two important qualifications to this negative result. First, the networks investigated all come from large, open communities and are incomplete. For incomplete networks the structural balance principle also permits nonclusterable sign configurations. Second, all three studies find that consistently with the principle of structural balance $+++$ and $+--$ triads occur more frequently than one would expect by random chance, and that $++-$ triads are underrepresented, but that evidence with respect to $---$ triads is ambiguous. These empirical studies thereby favor generalized balance over structural balance. The theoretical results obtained in this article predict that under a generalized balance regime balancing processes should lead complete networks to display sentiment configurations that permit a partition into multiple, possibly many antagonistic friendship cliques. The direction for future research this suggests is testing whether a tendency toward such group configurations of sentiment can be observed in close-knit communities.

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APPENDIX: PROOFS OF THEOREMS 1 AND 2

Theorem 1. *Under both a structural balance regime as well as a generalized balance regime, for* $m = n-1$ *and* $X = X^N$*, only balanced graphs are stable, and for any imbalanced graph* $G^1 = (N, X^N, F^1)$ *there exists an improving path* $(\{G^Z\}, \{i^{Z-1}\})$ *such that* G^Z *is balanced.*

Proof. We first note that if $G = (N, X^N, F)$ is balanced it must be stable because for any strategy $S_i \neq F_i$ generating the graph $G' = (N, X^N, (F \backslash F_i) \cup S_i)$ it holds that $\sum_j t(G, j) = 0 \Rightarrow t(G, i) \leq t(G', i) \Rightarrow u_i(G) \geq u_i(G')$.

Now consider any imbalanced graph $G^z = (N, X^N, F^z)$. Assume that $(\{G^z\}, \{i^{z-1}\})$ is an improving path of length z with $1 \leq z < Z$. Consider any two actors i,j such that $t(G^z, ij) > 0$ and $t(G^z, i) \geq t(G^z, j)$. Construct $G^{z+1} = (N, X^N, F^{z+1})$ such that $S^z = (F^z \backslash F^{z+1}) \cup (F^{z+1} \backslash F^z)$ is a strategy on G^z for actor i, $ij \in F^{z+1}$, and i and j are sign-equivalent on G^{z+1}. By sign-equivalence of i and j on G^{z+1} and $ij \in F^{z+1}$, it holds that $t(G^{z+1}, i) = t(G^{z+1}, j)$ and $t(G^{z+1}, ij) = 0 < t(G^z, ij)$. By S^z being a strategy on G^z for i it holds that $t(G^{z+1}, j) - t(G^{z+1}, ij) = t(G^z, j) - t(G^z, ij)$. It follows that $t(G^{z+1}, j) < t(G^z, j)$. From $t(G^z, i) \geq t(G^z, j)$, $t(G^{z+1}, i) = t(G^{z+1}, j)$, and $t(G^{z+1}, j) < t(G^z, j)$ it follows that $t(G^{z+1}, i) < t(G^z, i)$. Therefore, $(\{G^{z+1}\}, \{i^z\})$ of length $z+1$ with $i^z = i$ is also an improving path, hence G^z is not stable.

We have established that if G^z is imbalanced and $(\{G^z\}, \{i^{z-1}\})$ is an improving path then we can construct G^{z+1} such that $(\{G^{z+1}\}, \{i^z\})$ for some i^z is also an improving path. For $z = 1$, $\{G^z\}$ is simply the focal graph and an empty actor set so that the definition of improving path is trivially satisfied. By induction, $(\{G^Z\}, \{i^{Z-1}\})$ is an improving path.

By S^z being a strategy on G^z for i, it holds that $\sum_k t(G^{z+1}, k) - 3t(G^{z+1}, i) = \sum_k t(G^z, k) - 3t(G^z, i)$. It follows that $\sum_k t(G^{z+1}, k) < \sum_k t(G^z, k)$ for any z. Hence, for some finite Z, $\sum_k t(G^Z, k) = 0 \Rightarrow G^Z$ is balanced.

Theorem 2. *Under both a structural balance regime as well as a generalized balance regime, given any actor set* N *with cardinality* n *and given any edge set* $X \subseteq X^N$*, for any maximum number of simultaneous sign changes* $m > 0$*, only balanced graphs* $G = (N, X, F)$ *can be stochastically stable.*

Proof. A function $\rho()$ is an *ordinal potential function* for the game defined above if for any G and any strategy S_i it holds that sign $(\rho(G) - \rho(N, X, (F \backslash F_i) \cup S_i)) = \text{sign}(u_i(G) - u_i(N, X, (F \backslash F_i) \cup S_i))$. If a game has an ordinal potential function, then the set of stochastically stable outcomes is known to contain only outcomes with highest potential (Foster and Young, 1990; Blume, 1993; Young, 1998; Baron, Durieu, Haller, and Solal, 2006). Consider any $G = (N, X, F)$, any strategy S_i of any actor i on G generating $G' = (N, X, (F \backslash F_i) \cup S_i))$, and the function $\rho(G) = -\sum_j t(G, j)/3$. Now we have:

$$\begin{aligned}
\rho(G') - \rho(G) &= \left[\sum_j t(G,j)/3\right] - \left[\sum_j t(G',j)/3\right] \\
&= \left[t(G,i) + \left(\sum_j t(G,j)/3 - t(G,i)\right)\right] \\
&\quad - \left[t(G',i) + \left(\sum_j t(G',j)/3 - t(G',i)\right)\right] \\
&= \left[t(G,i) + \left(\sum_j t(G,j)/3 - t(G,i)\right)\right] \\
&\quad - \left[t(G',i) + \left(\sum_j t(G,j)/3 - t(G,i)\right)\right] \\
&\text{by } S_i \text{ being a strategy} \\
&= t(G,i) - t(G',i) \Rightarrow \\
\operatorname{sign}(\rho(G') - \rho(G)) &= \operatorname{sign}(u_i(G') - u_i(G)).
\end{aligned}$$

□

Hence, $\rho(G)$ is an ordinal potential function for the given game. $\rho(G)$ is maximal only if $\sum_j t(G, j) = 0$, namely whenever G is balanced. Therefore, G can be stochastically stable only if it is balanced.