

## Chapter 1. Review of Fundamentals

After elaborating a bit on the title and contents of the course, this short introductory chapter lists the basic notions and facts of the classical mechanics, which are supposed to be known to students from their undergraduate studies. Due to this reason, the explanations (if any) are very brief. For the remedial reading, virtually any undergraduate textbook on classical mechanics may be used.<sup>1</sup>

### 1.1. Mechanics and dynamics

This course covers the basics of both classical *mechanics* and *dynamics*. Though much intertwined, these notions are still different. Theoretical mechanics deals with deriving equations of motion (most frequently, ordinary or partial differential equations) of point-like particles and their systems (including solids and fluids), the solution of these equations, and interpretation of their results. For dynamics, there are various definitions; three of them may be met most frequently:

- (i) the part of mechanics which deals with motion (in contrast with *statics*);
- (ii) the part of mechanics which deals with reasons for motion (in contrast with *kinematics*);
- (iii) the part of mechanics which focuses on the two last parts of tasks of mechanics as a whole, i.e., the solution of the equations of motion and discussion of the results.

In my discussion, I will stick to the last definition which certainly invites (at least:-) two questions. First, it may look like mechanics and dynamics are just two sequential steps of a single process; why should they be considered separate disciplines? The main reason is that the many differential equations of motion, obtained in classical mechanics, also describe processes in different physical (and sometimes not only physical!) systems, so that their analysis may reveal important features of these systems as well. For example, the famous ordinary differential equation

$$\ddot{x} + \omega_0^2 x = 0, \quad (1.1)$$

where each dot denotes differentiation over time, may describe sinusoidal (“harmonic”) oscillations not only of a coordinate of a mass hung on a spring, but also the time evolution of an arbitrary spatial mode of electromagnetic waves in a linear medium. Similarly, the partial differential equation

$$\left( \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) f(\mathbf{r}, t) = 0, \quad (1.2)$$

where  $v$  is wave velocity, and  $\nabla^2$  is the Laplace operator,<sup>2</sup> may describe not only linearly polarized acoustic waves in a mechanical continuum (solid or fluid), but also electromagnetic waves in a non-

<sup>1</sup> See, for example (in the alphabetical order):

- G. R. Fowles and G. L. Cassiday, *Analytical Mechanics*, 7<sup>th</sup> ed., Brooks Cole, 2004;

- K. R. Symon, *Mechanics*, 3<sup>rd</sup> ed., Addison-Wesley, 1971;

- J. B. Marion and S. T. Thornton, *Classical Dynamics of Particles and Systems*, 4<sup>th</sup> ed., Saunders, 1995.

<sup>2</sup> My lecture series assumes student’s familiarity with the basic calculus and vector algebra. The most important (but not too trivial) formulas are listed in the appendix *Selected Mathematical Formulas* (referred to as MA). For example, see MA Sec. 9 for a reminder of the definition and the basic properties of the Laplace operator.

dispersive media, some chemical reactions, etc. Thus the results of their analysis may be “recycled” for applications well beyond mechanics.

The second natural question is that definition (iii) of dynamics is suspiciously close to mathematics, especially the differential equation analysis; what is the difference? To answer it, we have to dip, for just a second, into the philosophy of physics. Physics may be described as an art (and a bit of science :-)) of description of nature by mathematics; hence in many cases the approaches of a mathematician and a physicist to the same problem are very similar. The main difference is that the physicist tries to express the result of equation analysis in terms of the *system motion* rather than *function properties*, and as a result develop some sort of intuition (“gut feeling”) about how other, apparently similar, systems may behave, even if their equations of motion are somewhat different or not known at all. The intuition so developed has an enormous heuristic power, and most discoveries in physics have been made through gut-feeling-based insights rather than just plugging one equation into another.

Inside mechanics, term “dynamics” is sometimes also used in a somewhat different sense, as a study of causes of motion – forces and the like, i.e. as an alternative to *kinematics* – study of mechanical motion as such, regardless of its reasons. Let us start our brief review from kinematics.

## 1.2. Kinematics: Basic notions

The basic notions of kinematics may be defined in various ways, and mathematics pays a lot of attention to analyzing such systems of axioms and relations between them. In physics, we typically stick to less rigorous ways, and end debating a definition as soon as everybody in the room agrees that we are all speaking about the same thing. Let me hope that the following notions used in classical mechanics, do satisfy this criterion:

- (i) Geometric *point* (as a mathematical abstraction for a very small object).
- (ii) *Orthogonal, linear* (“Cartesian”) *coordinates* of the point in a particular “*coordinate system*” (alternatively called the *reference frame*) – see Fig. 1.<sup>3, 4</sup>

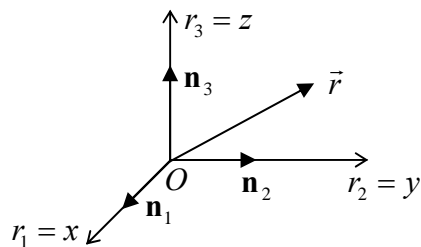


Fig. 1.1. Cartesian coordinates and radius-vector of a point/particle.

<sup>3</sup> In these notes the Cartesian coordinates are denoted either as  $\{r_1, r_2, r_3\}$  or as  $\{x, y, z\}$ , depending on convenience in the particular case. Note that axis numbering is important for operations like vector products; the “correct” (meaning generally accepted) numbering order is such that rotation  $\mathbf{n}_1 \rightarrow \mathbf{n}_2 \rightarrow \mathbf{n}_3 \rightarrow \mathbf{n}_1 \dots$  looks counterclockwise if watched from inside the positive quadrant (in which all  $r_i > 0$ ).

<sup>4</sup> In references to figures, formulas, problems and sections within the same chapter of these notes, the chapter number is dropped. References to other parts of the lecture notes series (whose draft versions are available on my Web site, <http://rsfq1.physics.sunysb.edu/~likharev/personal/>) use the following abbreviations: SM for *Statistical Mechanics*, EM for *Classical Electrodynamics*, and QM for *Quantum Mechanics*. In turn, this part of the notes is referred to as CM.

The coordinates may be used to define point's *radius-vector*

$$\mathbf{r}(t) = \sum_{j=1}^3 \mathbf{n}_j r_j(t), \quad (1.3)$$

where  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are unit vectors along coordinate axis directions.

(iii) (Instant) *velocity* of the point, as measured in the same reference frame,

$$\mathbf{v}(t) \equiv \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} \equiv \sum_{j=1}^3 \mathbf{n}_j \dot{r}_j(t), \quad (1.4)$$

and its *acceleration*:

$$\mathbf{a}(t) \equiv \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \ddot{\mathbf{r}}. \quad (1.5)$$

(iv) *Frame-to-frame transfer*. Since the above definitions of vectors  $\mathbf{r}$ ,  $\mathbf{v}$ , and  $\mathbf{a}$  depend on the chosen reference frame (are “reference-frame-specific”), we need to relate the vectors observed in different frames. For two reference frames, with the corresponding axes parallel in the moment of interest (Fig. 2), the relation between the radius-vectors is simple:

$$\mathbf{r}|_{\text{in } O'} = \mathbf{r}|_{\text{in } O} + \mathbf{r}_O|_{\text{in } O'}. \quad (1.6)$$

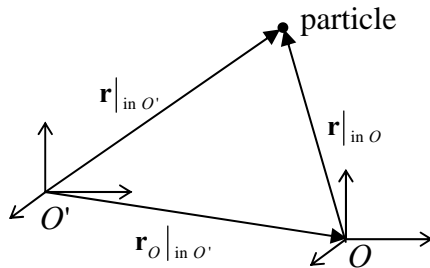


Fig. 1.2. Coordinate transfer between two reference frames.

If the frames move versus each other by *translation* only (no mutual rotation), similar relations are valid for velocity and acceleration as well:

$$\mathbf{v}|_{\text{in } O'} = \mathbf{v}|_{\text{in } O} + \mathbf{v}_O|_{\text{in } O'}, \quad (1.7)$$

$$\mathbf{a}|_{\text{in } O'} = \mathbf{a}|_{\text{in } O} + \mathbf{a}_O|_{\text{in } O'}. \quad (1.8)$$

However, in the case of mutual rotation of the reference frames, notions like  $\mathbf{v}_O|_{\text{in } O'}$  are not well defined. (Indeed, different points of a rigid body connected to frame  $O'$  may have different velocities in frame  $O$ .) As a result, the transfer laws for velocities and accelerations are more complex than those given by Eqs. (8) and (9). It will be more natural for us to discuss them in the end of Chapter 5 which is devoted to a rigid body motion.

(v) *Particle*: a physical body whose size is negligible, and shape unimportant for the given problem.<sup>5</sup> Since classical mechanics neglects the quantum effects,<sup>6</sup> particle's position, at any particular instant  $t$ , may be identified with a single geometric point, i.e. one radius-vector (3). Actually, using Eq. (3) implies that the space has the *Euclidean metric*:

$$r^2 = \sum_{j=1}^3 r_j^2. \quad (1.9)$$

which is independent, in particular, of the mass distribution in space. By sticking to Eq. (9), and to an implicit assumption that time  $t$  runs similarly in all reference frames, we subscribe to the notion of the *absolute* (“Newtonian”) *space/time*, and hence abstain from discussion of any relativistic effects in this course.<sup>7</sup>

Finding the *laws of motion*  $\mathbf{r}(t)$  of all particles participating in the given problem may be considered the final goal of classical mechanics.

### 1.3. Dynamics: Newton laws

Generally, classical dynamics is completely described (in addition to the kinematic relations given above) by three *Newton laws*.<sup>8</sup> In contrast to the impression some textbooks try to create, these laws are experimental in nature, and cannot be derived from purely theoretical arguments.<sup>9</sup>

I am confident that the reader is familiar with Newton laws, in one or another formulation. Let me note only that in some formulations the 1<sup>st</sup> Newton law looks just as a particular case of the 2<sup>nd</sup> law (for the case of zero net force). In order to avoid this duplication, the 1<sup>st</sup> law may be postulated as the *existence* of at least one reference frame (called *inertial*) in which any free particle (isolated from the rest of the Universe) moves with  $\mathbf{v} = \text{const}$ , i.e. with  $\mathbf{a} = 0$ . According to Eq. (8), this postulate immediately implies that there is also an infinite number of other inertial frames, because all frames  $O'$  moving without rotation or acceleration relative to the initial inertial frame  $O$  (i.e. having  $\mathbf{a}_O|_{\text{in } O'} = 0$ ) are also inertial.

<sup>5</sup> The last qualification is extremely important. For example, the size and shape of a space shuttle are not too important for the discussion of its orbital motion, but are paramount when its landing procedures are developed.

<sup>6</sup> This approximation is legitimate, crudely, when the product of scales of coordinate and momentum of the particle motion, along each coordinate of interest, is much larger than the Planck's constant  $\hbar \approx 1.054 \times 10^{-34}$  J·s. For a more exact formulation, the reader is referred to the Quantum Mechanics (“QM”) part of these notes. More exact values of  $\hbar$  and some other physical constants may be found in a list available on our course's Web site.

<sup>7</sup> Following tradition, an introduction to special relativity is included into the Classical Electrodynamics (“EM”) part of my lecture course series. Relativistic mechanics shows that the relativistic effects are small if all particles velocities are much lower than the speed of light,  $c \approx 3.00 \times 10^8$  m/s, and all distances are much larger than the system's Schwarzschild radius  $r_s \equiv 2Gm/c^2$ , where  $G \approx 6.8 \times 10^{-11}$  m<sup>3</sup>/kg·s<sup>2</sup> is the Newtonian gravity constant. (See the *Physics Constants* appendix for more exact values, and some discussion, of the fundamental constants.)

<sup>8</sup> Due to the genius of Sir Isaac Newton (1643 - 1727), these laws were formulated as early as in 1687, far ahead of the science of that time.

<sup>9</sup> Some laws of Nature may be either postulated or derived from certain more general postulates, such as the Hamilton (“least action”) principle (see Sec. 10.2 below), but only if all their corollaries comply with all experimental results (or, more importantly, do not contradict to a single reliable experimental fact).

On the other hand, the 2<sup>nd</sup> and 3<sup>rd</sup> Newton laws may be postulated *together* in the following elegant way. Each particle, say with number  $k$ , may be characterized by a scalar constant (called *mass*  $m_k$ ), such that at any interaction of  $N$  particles (isolated from the rest of the Universe), in any inertial system,

$$\mathbf{P} \equiv \sum_{k=1}^N m_k \mathbf{v}_k = \text{const.} \quad (1.10)$$

(Each component of this sum,

$$\mathbf{p}_k \equiv m_k \mathbf{v}_k, \quad (1.11)$$

is called the *mechanical momentum* of the corresponding particle, and the whole sum  $\mathbf{P}$ , the *full momentum* of the system.)

Let us apply this postulate to the case of just two interacting particles. Differentiating Eq. (10), written for this case, over time, we get

$$\dot{\mathbf{p}}_1 = -\dot{\mathbf{p}}_2. \quad (1.12)$$

Let us give the derivative  $\dot{\mathbf{p}}_1$  the name of *force*  $\mathbf{F}_1$  exerted on particle 1. In this case, when its only possible source of force is particle 2, it may be denoted as  $\mathbf{F}_{12}$ . Similarly,  $\mathbf{F}_{21} \equiv \dot{\mathbf{p}}_2$ , so that we get the 3<sup>rd</sup> Newton law

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \quad (1.13)$$

Now, returning to the general case of several interacting particles, we see that an additional (but very natural) assumption that all partial forces  $\mathbf{F}_{kk'}$  acting on particle  $k$  add up, leads to the general form of the 2<sup>nd</sup> Newton law<sup>10</sup>

$$m_k \mathbf{a}_k = \dot{\mathbf{p}}_k = \sum_{k' \neq k} \mathbf{F}_{kk'} \equiv \mathbf{F}_k. \quad (1.14)$$

Equation (14) may be naturally generalized to the case when a system of  $N$  interacting particles is affected by external forces, so that

$$\mathbf{F}_k = \mathbf{F}_k^{(\text{ext})} + \sum_{k=1}^N \mathbf{F}_{kk'}. \quad (1.15)$$

If we sum up the resulting Eqs. (14) for all particles of the system then, due to the 3<sup>rd</sup> Newton law (13), the contributions of all internal forces to this double sum in the right-hand part cancel, and we get the equation

$$\dot{\mathbf{P}} = \mathbf{F}^{(\text{ext})}, \quad \mathbf{F}^{(\text{ext})} \equiv \sum_{k=1}^N \mathbf{F}_k^{(\text{ext})}, \quad (1.16)$$

which tells us that the translational motion of the system as the whole is similar to that of a single particle, under the effect of the *net external force*. In particular (as a sanity check), in the absence of external forces we return to postulate (10).

<sup>10</sup> Of course, for bodies of varying mass (e.g., rockets emitting jets),  $\dot{\mathbf{p}} \neq m\mathbf{a}$ , and the 2<sup>nd</sup> Newton law is only valid for  $\dot{\mathbf{p}}$  - see, e.g., Problem 1.

As a matter of principle, if the dependence of all pair forces  $\mathbf{F}_{kk'}$  of particle positions (and generally maybe of time as well) is known, the Newton laws, augmented by equations of (5), (6) of kinematics, allow the calculation of the laws of motion  $\mathbf{r}_k(t)$  of all particles of the system. For example, for one particle the 2<sup>nd</sup> law (14) gives the ordinary differential equation of the second order,

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t), \quad (1.17)$$

which may be integrated – either analytically or at least numerically. For certain cases, this is very simple. As an elementary example, for the motion of a particle in gravitational field of the Earth near its surface, an acceptable approximation to the Newton’s gravity law is

$$\mathbf{F} = m\mathbf{g}, \quad (1.18)$$

with  $\mathbf{g}$  being constant and directed down (toward Earth’s center), and mass  $m$  is the same as in Eq. (17).<sup>11</sup> As a result, mass cancels, and Eq. (17) becomes just  $\ddot{\mathbf{r}} = \mathbf{g}$  and may be easily integrated twice:

$$\dot{\mathbf{r}}(t) \equiv \mathbf{v}(t) = \int_0^t \mathbf{g} dt' + \mathbf{v}(0) = \mathbf{g}t + \mathbf{v}(0), \quad (1.19a)$$

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(t') dt' + \mathbf{r}(0) = \mathbf{g} \frac{t^2}{2} + \mathbf{v}(0)t + \mathbf{r}(0). \quad (1.19b)$$

(Each of these equations describes time evolution of three scalar components of vectors  $\mathbf{v}$  and  $\mathbf{r}$ .)

All this looks (and indeed is) very simple, but in most other cases leads to more complex calculations. Let us consider another simple problem: a bead of mass  $m$  sliding, without friction, along a round ring of radius  $R$  in a gravity field obeying Eq. (18) – see Fig. 3. Suppose we are only interested in bead’s velocity  $v$  in the lowest point, after it has been dropped from the rest at the rightmost position. If we want to solve this problem using only the Newton laws, we have to do pass through the following steps:

(i) consider the bead in an arbitrary intermediate position on a ring, described, for example by angle  $\theta$  (Fig. 3);

(ii) draw all the forces acting on the particle, in our case the gravity force  $m\mathbf{g}$ , and the reaction force  $\mathbf{N}$  exerted by the ring;

(iii) write the 2<sup>nd</sup> Newton law for two nonvanishing components of the bead acceleration, say for its vertical and horizontal components  $a_x$  and  $a_y$ ;

(iv) comprehend that in the absence of friction, force  $\mathbf{N}$  should be normal to the ring, so that we can use two additional equations,  $N_x = -N \sin\theta$  and  $N_y = N \cos\theta$ ;

(v) exclude unknown variables  $N$ ,  $N_x$ , and  $N_y$  from the resulting system of 4 equations, thus getting a single differential equation for one variable, for example  $\theta$ ;

<sup>11</sup> The last fact, the so-called “weak equivalence principle”, is highly nontrivial, but is verified experimentally to the accuracy of at least  $10^{-12}$ . Due to the importance of the principle, its much more sensitive (to  $\sim 10^{-18}$ ) tests are being planned – see, e.g., <http://www.sstd.rl.ac.uk/fundphys/step/>.

(vi) integrate this equation once to get the expression relating velocity  $\dot{\theta}$  and angle  $\theta$ ; and, finally

(vii) using our specific initial condition ( $\dot{\theta} = 0$  at  $\theta = \pi/2$ ), find the final velocity as  $v = R\dot{\theta}$  at  $\theta = 0$ .

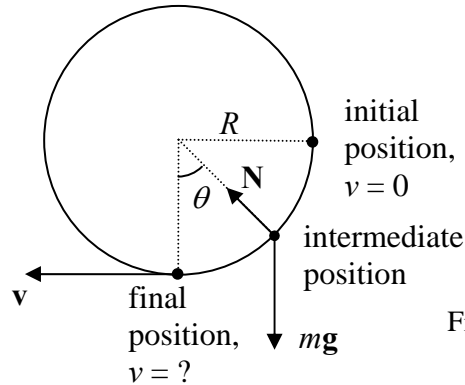


Fig. 1.3. Bead sliding down a vertical ring.

This is all very much doable, but cumbersome, for such a simple problem. In many other cases, writing equations of motion along relevant coordinates is very complex, and any help the general theory may provide is highly valuable. In many cases, such help is given by *conservation laws*; let us review the most general of them.

#### 1.4. Conservation laws

(i) Momentum. The first such law, the conservation of the full momentum of any system of particles isolated from the rest of the world, has already been discussed – see Eq. (10). In the case of one free particle the law is reduced to a trivial equation  $\mathbf{p} = \text{const}$ , i.e.  $\mathbf{v} = \text{const}$ . Just a reminder: this result is only valid in an inertial reference frame - because the Newton laws are only valid in such frames.

(ii) Angular momentum of a particle is defined as

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}, \quad (1.20)$$

where  $\mathbf{a} \times \mathbf{b}$  means the *vector* (“cross-“) product of the operands – see, e.g., MA Eq. (7.3). Now, differentiating Eq. (20) over time, we get

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}}. \quad (1.21)$$

In the first product,  $\dot{\mathbf{r}}$  is just the velocity vector  $\mathbf{v}$  which is parallel to the particle momentum  $m\mathbf{v}$ , so that this product vanishes, since the vector product of two parallel vectors is zero. In the second product,  $\dot{\mathbf{p}}$  equals the full force  $\mathbf{F}$  acting on the particle, so that Eq. (21) is reduced to

$$\dot{\mathbf{L}} = \boldsymbol{\tau}, \quad (1.22)$$

where vector

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}, \quad (1.23)$$

is called *torque*. (It is clearly reference-frame specific! And again, the frame has to be inertial for Eq. (22) to be valid!) For an important particular case of a *central* force  $\mathbf{F}$ , which is parallel to the radius

vector  $\mathbf{r}$  of a particle (measured from the force source point), the torque vanishes, so that the angular momentum is conserved:

$$\mathbf{L} = \text{const.} \quad (1.24)$$

(The same result is valid, of course, if there no force at all; then both Eqs. (10) and (24) are valid.)

For a system of  $N$  particles, the full angular momentum is naturally defined as

$$\mathbf{L} \equiv \sum_{k=1}^N \mathbf{L}_k. \quad (1.25)$$

Differentiating this equation over time, using Eq. (22) for each  $\dot{\mathbf{L}}_k$ , and again partitioning each force in a accordance with Eq. (15), we get

$$\dot{\mathbf{L}} = \sum_{\substack{k,k'=1 \\ k' \neq k}}^N \mathbf{r}_k \times \mathbf{F}_{kk'} + \boldsymbol{\tau}^{(\text{ext})}, \quad \boldsymbol{\tau}^{(\text{ext})} \equiv \sum_{k=1}^N \mathbf{r}_k \times \mathbf{F}_k^{(\text{ext})}. \quad (1.26)$$

The first (double) sum may be always broken into pairs of the type  $\mathbf{r}_k \times \mathbf{F}_{kk'} + \mathbf{r}_{k'} \times \mathbf{F}_{k'k}$ . Each of these pairs equals zero. Indeed, both components of the pair are vectors perpendicular to the plane to which both particles and the reference frame origin belong (Fig. 4). Also, due to the 3<sup>rd</sup> Newton law the two forces are equal and opposite, and each term in the sum may be presented as  $|F_{kk'}| h_{kk'}$ , with equal ‘‘lever arms’’  $h_{kk'} = h_{k'k}$ . As a result, the whole double sum in Eq. (26) vanishes, and we are left with a very simple result

$$\dot{\mathbf{L}} = \boldsymbol{\tau}^{(\text{ext})} \quad (1.27)$$

which is similar to Eq. (22) for a single particle, and is the angular analog of Eq. (16). If the full external torque  $\boldsymbol{\tau}^{(\text{ext})}$  by some reason vanishes (e.g., the system of particles is isolated from the rest of the Universe), the conservation law (24) is valid for the full angular momentum, even if its individual components  $\mathbf{L}_k$  are not conserved.

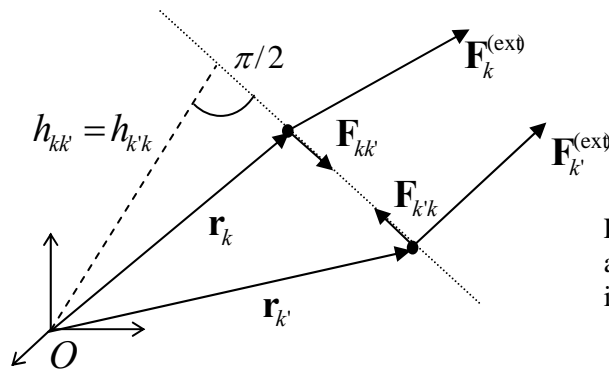


Fig. 1.4. Internal and external forces, and the internal torque cancellation in a system of two particles.

(iii) Energy conservation is arguably the most general physics law, but in mechanics it takes a more humble form of *mechanical energy conservation*, and has limited applicability - just as the two conservation laws discussed above. To derive it, we first define *kinetic energy* of a particle as

$$T \equiv \frac{m}{2} v^2, \quad (1.28)$$

and then notice that its change at any path from point  $A$  to point  $B$  always obeys the following relation:

$$\Delta T \equiv T(\mathbf{r}_B) - T(\mathbf{r}_A) = \int_A^B \mathbf{F} \cdot d\mathbf{r}, \quad (1.29)$$

where  $\mathbf{F}$  is the full force acting on the particle, and symbol  $\mathbf{a} \cdot \mathbf{b}$  denotes the *scalar* product of vectors  $\mathbf{a}$  and  $\mathbf{b}$  – see, e.g., MA Eq. (7.1). In order to prove Eq. (29), it is sufficient to consider the differential

$$dT \equiv d\left(\frac{m}{2}v^2\right) = d\left(\frac{m}{2}\mathbf{v} \cdot \mathbf{v}\right) = m\mathbf{v} \cdot d\mathbf{v} = m\frac{d\mathbf{r}}{dt} \cdot d\mathbf{v} = m\frac{d\mathbf{r} \cdot d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt} \cdot d\mathbf{r}. \quad (1.30)$$

Now, using the 2<sup>nd</sup> Newton Law,  $d\mathbf{p}/dt = \mathbf{F}$ , we get relation  $dK = \mathbf{F} \cdot d\mathbf{r}$  whose integration from  $A$  to  $B$  gives Eq. (29). The integral in the RHP of that equation is called *work* of force  $\mathbf{F}$  on the path from  $A$  to  $B$ .

The further step may be made only for *potential* (also called *conservative*) forces which may be presented as gradients of some scalar function  $U(\mathbf{r})$ , called the *potential energy*.<sup>12</sup> Operator del allows a very compact expression of this fact:

$$\mathbf{F} = -\nabla U. \quad (1.31)$$

For example, for the uniform gravity field (18),

$$U = mgh + \text{const}, \quad (1.32)$$

where  $h$  is the vertical coordinate directed up.

Integrating the tangential component of the vector  $\mathbf{F}$ , given by Eq. (31), along the path from  $A$  to  $B$ , we get

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = U(\mathbf{r}_A) - U(\mathbf{r}_B), \quad (1.33)$$

i.e. work of potential forces may be presented as the difference of values of function  $U(\mathbf{r})$  in the initial and final point of the path.

As a parenthetic remark, according to Eq. (33), work of a potential force on any closed trajectory is zero, because for any choice of points  $A$  and  $B$  on such path (Fig. 5),<sup>13</sup>

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r} \Big|_{\text{path 1}} + \int_B^A \mathbf{F} \cdot d\mathbf{r} \Big|_{\text{path 2}} = [U(\mathbf{r}_A) - U(\mathbf{r}_B)] + [U(\mathbf{r}_B) - U(\mathbf{r}_A)] = 0. \quad (1.34)$$

<sup>12</sup> Note that because of its definition via the gradient, the potential energy is only defined to an arbitrary additive constant.

<sup>13</sup> This result may be also obtained in a (formally) different way. According to the Stokes theorem (see MA Eq. (12.1)), for any continuous vector, in particular the force, we may write

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_A (\nabla \times \mathbf{F})_n d^2r,$$

where  $A$  is the area of any surface bound by the closed contour  $C$  we are considering. But the vector analysis says that the curl of any vector-gradient of the type (31) equals zero,  $\nabla \times \mathbf{F} = 0$  – see, e.g., MA Eq. (11.1). Thus we immediately arrive at Eq. (34).

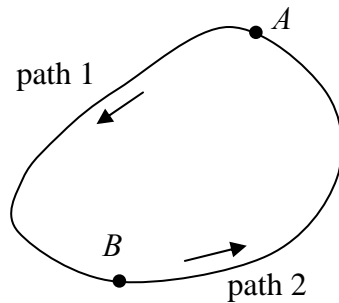


Fig. 1.5. Work of a force on a closed path.

Now, combining Eqs. (29) and (33) we see that

$$T(\mathbf{r}_B) - T(\mathbf{r}_A) = U(\mathbf{r}_A) - U(\mathbf{r}_B), \quad (1.35)$$

so that the *total mechanical energy* defined as

$$E \equiv T + U \quad (1.36)$$

is indeed conserved:

$$E(\mathbf{r}_A) \equiv T(\mathbf{r}_A) + U(\mathbf{r}_A) = T(\mathbf{r}_B) + U(\mathbf{r}_B) \equiv E(\mathbf{r}_B), \quad (1.37)$$

but for conservative forces only. (Non-conservative forces, e.g., friction, typically transfer energy from mechanical form into some other form, e.g., heat.)

The mechanical energy conservation allows us to solve this problem shown in Fig. 3 in one shot by writing Eq. (37) for the initial and final points:<sup>14</sup>

$$0 + mgR = \frac{m}{2}v^2 + 0. \quad (1.38)$$

Solving Eq. (38) for  $v$  immediately gives as the final answer. The reader has to agree that this way is much more effective. From the mathematical point of view, most conserved quantities present *the first integrals of motion* which liberate us from the necessity to integrate the second-order differential equations of motion, following from the Newton laws, twice.

### 1.5. OK, can we go home now?

Not yet. In many cases the conservation laws discussed above provide little help. Consider for example a generalization of the bead-on-the-ring problem shown in Fig. 3, in which the ring is rotated by external forces, with a constant angular velocity  $\omega$ , about its vertical diameter (Fig. 6).

In this problem (which will be repeatedly used below, as an analytical mechanics testbed), none of the three conservation laws listed in the last section, holds. In particular, bead's energy

$$E = \frac{m}{2}v^2 + mgh \quad (1.39)$$

is *not* constant, because the external forces rotating the ring may change it. Of course, we still can solve the problem using the Newton laws, but this is even more complex than for the case of ring at rest, in

<sup>14</sup> Here the arbitrary constant in Eq. (32) is chosen so that the potential energy is zero in the finite point.

particular because the force  $\mathbf{N}$  exerted on the bead by the ring now may have three rather than two Cartesian components which are not easily related.

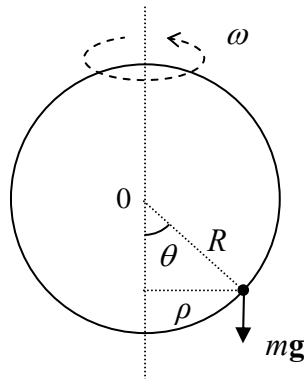


Fig. 1.6. Bead on a rotating ring.

One can readily see that if we could exclude the “reaction forces” such as  $\mathbf{N}$ , which ensure *external constraints* of the particle motion, in advance, that would help a lot. Such an exclusion may be provided by analytical mechanics, in particular its Lagrangian formulation, to which we now proceed.