

## Chapter 2. Lagrangian Formalism

The goal of this chapter is to describe the Lagrangian approach to classical mechanics, which is extremely useful for obtaining the differential equations of motion (and sometimes their first integrals) not only for mechanical systems with holonomic constraints, but also other dynamic systems.

### 2.1. Lagrange equations

In many cases, constraints imposed on 3D motion of a system of  $N$  particles may be described by  $N$  vector (i.e. 3 scalar) equations

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, \dots, q_j, \dots, q_J, t), \quad 1 \leq k \leq N, \quad (2.1)$$

where  $q_j$  are *generalized* coordinates which (together with constraints) completely define the system position, and  $J \leq 3N$  is the number of the actual degrees of freedom.<sup>1</sup> For example, for our testbed, bead-on-rotating-ring problem (Fig. 1.6),  $J = 1$ , so that it may be described by one generalized coordinate – for example, the polar angle  $\theta$ . With the reference frame centered at the center of the ring, Eq. (1) has the form (cf. MA Eq. (10.6)):

$$\mathbf{r} = \{R \sin \theta \cos(\omega t + \varphi), R \sin \theta \sin(\omega t + \varphi), -R \cos \theta\}, \quad (2.2)$$

where constant  $\varphi$  depends on the selection of the time origin.

Now, let us consider a set of small *virtual displacements* (a.k.a. *variations*)  $\delta q_j$  allowed by the constraints. Virtual displacements differ from real infinitesimal displacements (described by *differentials*  $dq_j$ ) in that  $\delta q_j$  describes not the system's motion itself, but rather its possible variation – see Fig. 1. However, in most respects, mathematical operations with variations follow the same calculus rules as the usual differentials. For example, we can calculate the variation of Eq. (1) at fixed time  $t$ , using the general formula for the differentiation of a function of several arguments – see, e.g., MA Eq. (4.2):

$$\delta \mathbf{r}_k = \sum_j \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j. \quad (2.3)$$

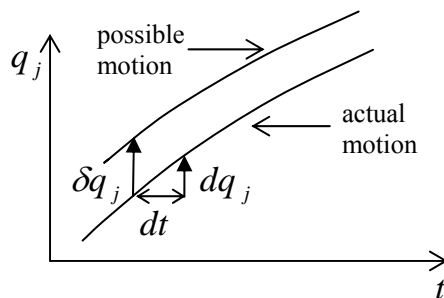


Fig. 2.1. Displacement  $dq_j$  and variation  $\delta q_j$ .

<sup>1</sup> The constraints that allow such description are called *holonomic*. Possibly, the simplest example of a non-holonomic constraint is a set of hard walls constraining the motion of gas molecules in a closed volume, which may be describe by an inequality rather than equality. Non-holonomic constraints are better dealt with other methods, e.g., by imposing boundary conditions on (otherwise unconstrained) motion.

Now let us break the forces acting upon each particle into two parts: frictionless parts  $\mathbf{N}_k$  of the constraining forces and “external” forces  $\mathbf{F}_k$ , including possible friction parts of the reaction forces. Then the 2<sup>nd</sup> Newton law for  $k$ -th particle of the system may be presented as

$$m_k \dot{\mathbf{v}}_k - \mathbf{F}_k = \mathbf{N}_k. \quad (2.4)$$

Since any variation of the motion has to be allowed by the constraints, its  $3N$ -dimensional vector with  $N$  3D vector components  $\delta\mathbf{r}_k$  has to be perpendicular to the  $3N$ -dimensional vector of force, also with  $N$  3D vector components,  $\mathbf{N}_k$ . (For example, for the problem shown in Fig. 1.3, the virtual displacement vector  $\delta\mathbf{r}_k$  may be directed only along the ring, while force  $\mathbf{N}$  exerted by the ring is perpendicular to that direction.) This condition may be expressed as

$$\sum_k \mathbf{N}_k \cdot \delta\mathbf{r}_k = 0, \quad (2.5)$$

where the scalar product of  $3N$ -dimensional vectors is defined exactly as that of 3D vectors, i.e. as the sum of the products of the corresponding components of the operands. Plugging Eq. (4) into Eq. (5), we get the so-called *D’Alembert* (or “Lagrange-D’Alembert”) *principle*:<sup>2</sup>

$$\sum_k (m_k \dot{\mathbf{v}}_k - \mathbf{F}_k) \cdot \delta\mathbf{r}_k = 0. \quad (2.6)$$

Plugging Eq. (3) into Eq. (6), this result may be presented as

$$\sum_{j,k} m_k \dot{\mathbf{v}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j - \sum_j \mathcal{F}_j \delta q_j = 0, \quad (2.7)$$

where we are using a very convenient notion of a *generalized force*<sup>3</sup>

$$\mathcal{F}_j \equiv \sum_k \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j}. \quad (2.8)$$

Now, let us transform Eq. (7), using two mathematical identities. First, using differentiation by parts to calculate the following time derivative:

$$\frac{d}{dt} \left( \dot{\mathbf{r}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} \right) = \dot{\mathbf{v}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} + \dot{\mathbf{r}}_k \cdot \frac{d}{dt} \left( \frac{\partial \mathbf{r}_k}{\partial q_j} \right), \quad (2.9)$$

we may notice that the first term in the right-hand part is exactly the product in the first term of Eq. (7). In addition, using the fact that the variation ( $\partial$ ) and differentiation ( $d$ ) are evidently interchangeable operations (Fig. 2), in the second term of Eq. (9) we can write

$$\frac{d}{dt} \left( \frac{\partial \mathbf{r}_k}{\partial q_j} \right) = \frac{\partial \dot{\mathbf{v}}_k}{\partial q_j}. \quad (2.10)$$

Second, let us differentiate of Eq. (1) over time:

<sup>2</sup> Actually, the core of this result was first obtained by J. Bernoulli (1667 – 1748).

<sup>3</sup> Note that since the dimensionality of generalized coordinates is arbitrary (in our bead-on-a-ring problem, it may be angle  $\theta$ ), that of generalized forces may also differ from  $[F]$ .

$$\mathbf{v}_k \equiv \frac{d\mathbf{r}_k}{dt} = \sum_j \frac{\partial \mathbf{r}_k}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t}. \quad (2.11)$$

This equation shows that particle velocities  $\mathbf{v}_k$  may be considered as linear functions of generalized velocities  $\dot{q}_j$ , with coefficients

$$\frac{\partial \mathbf{v}_k}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_k}{\partial q_j}. \quad (2.12)$$

(Of course, this does not prevent the velocities being also functions of the generalized coordinates and time. In the right-hand part of Eq. (11), these dependencies are hidden inside the partial derivatives of the first term, and in the last term.) If we accept this (very useful!) standpoint, then Eq. (7), with the account of Eqs. (9), (10), and (12), turns into

$$\sum_j \left\{ \frac{d}{dt} \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial \dot{q}_j} - \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial q_j} - \mathcal{F}_j \right\} \delta q_j = 0. \quad (2.13)$$

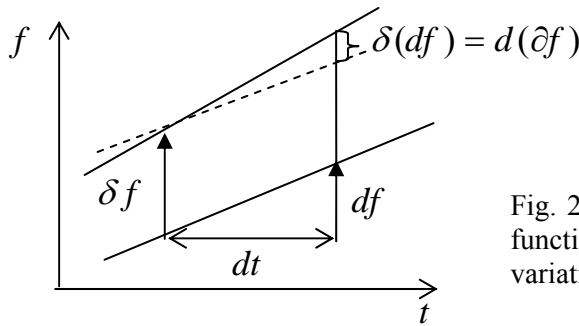


Fig. 2.2. A variation of a differential (of any function  $f$ ) is the same as the differential of its variation.

This result may be further simplified by applying a similar approach to the total kinetic energy of the system,

$$T \equiv \sum_k \frac{m_k}{2} v_k^2 = \frac{1}{2} \sum_k m_k \mathbf{v}_k \cdot \mathbf{v}_k. \quad (2.14)$$

Let us commit ourselves, from this point on, to considering  $T$  (just like  $\mathbf{v}_k$  above) a function of not only the generalized coordinates  $q_j$  and time, but also of the generalized velocities  $\dot{q}_i$  as *independent variables*. Then we can calculate its partial derivatives as

$$\frac{\partial T}{\partial q_j} = \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial q_j}, \quad \frac{\partial T}{\partial \dot{q}_j} = \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial \dot{q}_j}, \quad (2.15)$$

and notice that they are exactly the sums participating in Eq. (13).

Now comes the standard argument of the variational calculus: in order for the left-hand part of that equation to be zero for an arbitrary set of variations  $\delta q_j$ , each expression in the curly brackets should equal zero. This leads us to a system of  $J$  *Lagrange equations*,<sup>4</sup>

<sup>4</sup> Derived by Joseph-Louis Lagrange (1736-1813) who pioneered the whole field of analytical mechanics (not to mention his key contributions to number theory and celestial mechanics).

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - \mathcal{F}_j = 0, \quad \text{for } j = 1, 2, \dots, J. \quad (2.16)$$

This is as far as we can go for arbitrary forces. However, if all the forces may be expressed in the form similar but somewhat more general than Eq. (1.31),

$$\mathbf{F}_k = -\nabla_k U(r_1, r_2, \dots, r_N, t), \quad (2.17)$$

where  $U$  is the effective potential energy of the system,<sup>5</sup> and sign  $\nabla$  index  $k$  means the differentiation over coordinates of  $k$ -th particle, we can write

$$\mathcal{F}_j \equiv \sum_k \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} = -\sum_k \left( \frac{\partial U}{\partial x_k} \cdot \frac{\partial x_k}{\partial q_j} + \frac{\partial U}{\partial y_k} \cdot \frac{\partial y_k}{\partial q_j} + \frac{\partial U}{\partial z_i} \cdot \frac{\partial z_i}{\partial q_j} \right) \equiv -\frac{\partial U}{\partial q_j}. \quad (2.18)$$

Since the potential energy (17) depends only on coordinates but not velocities,  $\partial U / \partial \dot{q}_j = 0$ , each Lagrange equation (16), with the substitution of Eq. (18), may be presented in its “canonical” form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad \text{where } L \equiv T - U. \quad (2.19)$$

Using the Lagrangian formalism in practice, please remember that:

(i) Each system has only *one* Lagrange function  $L$ , but is described by  $J \geq 1$  Lagrange equations of motion (for  $j = 1, 2, \dots, J$ ).

(ii) Differentiating  $T$ , do not forget to consider generalized velocities  $\dot{q}_j$  as independent variables, forgetting for a while that they are the time derivatives of  $q_j$ .

## 2.2. Examples

As the first, simplest example, consider a particle constrained to move along one axis (say,  $x$ ):

$$T = m \frac{\dot{x}^2}{2}, \quad U = U(x, t). \quad (2.20)$$

In this case, it is natural to consider  $x$  as the (only) generalized coordinate, and  $\dot{x}$  as the corresponding velocity, so that

$$L \equiv T - U = \frac{m}{2} \dot{x}^2 - U(x, t). \quad (2.21)$$

Considering  $\dot{x}$  an independent variable, we get  $\partial L / \partial \dot{x} = m\dot{x}$ , and  $\partial L / \partial x = -\partial U / \partial x$ , so that the Lagrange equation of motion (only one equation in this case of the single degree of freedom!) yields

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<sup>5</sup> Note that due to the possible time dependence of  $U$ , Eq. (17) does not mean that forces  $\mathbf{F}_k$  have to be conservative – see the next section for more discussion. With this understanding, I will still use for  $U$  the convenient name of “potential energy”.

$$\frac{d}{dt}(mv) - \left(-\frac{\partial U}{\partial x}\right) = 0, \quad (2.22)$$

evidently the same result as the  $x$ -component of the 2<sup>nd</sup> Newton law. This is a good sanity check, but we see that the Lagrange formalism does not provide too much advantage in this particular case.

This advantage is, however, evident for our testbed problem (Fig. 1.6). Indeed, associating  $q$  with the polar angle  $\theta$ , we see that in this case the kinetic energy depends not only on the generalized velocity, but also on the generalized coordinate:<sup>6</sup>

$$T = \frac{m}{2}R^2(\dot{\theta}^2 + \omega^2 \sin^2 \theta), \quad L = T - U = \frac{m}{2}R^2(\dot{\theta}^2 + \omega^2 \sin^2 \theta) + mgR \cos \theta + \text{const.} \quad (2.23)$$

Here it is especially important to remember that at composing the Lagrange equation,  $\theta$  and  $\dot{\theta}$  have to be treated as independent variables, so that

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mR^2 \omega^2 \sin \theta \cos \theta - mgR \sin \theta, \quad (2.24)$$

giving the equation of motion

$$\frac{d}{dt}(mR^2 \dot{\theta}) - (mR^2 \omega^2 \sin \theta \cos \theta - mgR \sin \theta) = 0. \quad (2.25)$$

As a sanity check, at  $\omega \rightarrow 0$  we get the correct equation of the usual pendulum:

$$\ddot{\theta} + \Omega^2 \sin \theta = 0, \quad \Omega^2 \equiv \frac{g}{R}. \quad (2.26)$$

We will explore the full dynamic equation (25) in more detail later, but please Note how simple its derivation was, in comparison with writing the Newton laws and then excluding the reaction force.

Next, it is very important that though the Lagrangian formalism was derived for mechanical systems, the resulting equations are applicable to other dynamic systems, especially those for which the kinetic and potential energies may be readily expressed via some generalized coordinates. As the simplest example, consider the well-known system (Fig. 3) of a capacitor with capacitance  $C$  and an inductive coil with self-inductance  $L$ .<sup>7</sup> (Engineers frequently call it the “ $LC$  tank circuit”.)

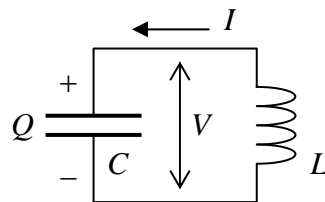


Fig. 2.3.  $LC$  tank circuit.

<sup>6</sup> This expression for  $T \equiv (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$  may be readily obtained either by differentiating Eq. (2) over time, or just by noticing that the velocity vector has two perpendicular components: one along the ring (with magnitude  $R\dot{\theta}$ ) and another one normal to the ring plane (with magnitude  $\omega\rho = \omega R \sin \theta$  - see Fig. 1.6).

<sup>7</sup> Hopefully, this traditional notation would not lead to the confusion between the inductance and the Lagrange function.

At relatively low frequencies (see, e.g., EM Sec. 6.2), we can use the well-known “lumped-model” expressions for the circuit component energies:

$$E_C = \frac{Q^2}{2C}, \quad E_L = \frac{LI^2}{2}. \quad (2.27)$$

Since current  $I$  and electric charge  $Q$  on the capacitor are connected by the continuity equation  $dQ/dt = I$ , it is natural to declare the charge a generalized coordinate, and the current, the generalized velocity. With this choice, the electrostatic energy  $E_C(Q)$  should be treated as the potential energy  $U$  of the system, and the magnetic energy  $E_L(I)$ , as its kinetic energy  $T$ .<sup>8</sup> With this attribution, we get

$$\frac{\partial T}{\partial \dot{q}} \equiv \frac{\partial E_L}{\partial I} = LI \equiv L\dot{Q}, \quad \frac{\partial T}{\partial q} \equiv \frac{\partial E_L}{\partial Q} = 0, \quad \frac{\partial U}{\partial q} \equiv \frac{\partial E_C}{\partial Q} = \frac{Q}{C}, \quad (2.28)$$

so that the Lagrangian equation of motion is

$$\frac{d}{dt}(L\dot{Q}) + \frac{Q}{C} = 0. \quad (2.29)$$

If both parameters of the circuit,  $L$  and  $C$ , are constant in time, this is just the harmonic oscillator equation similar to Eq. (1.1), and describes sinusoidal oscillations with frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}. \quad (2.30)$$

This is of course a very well known result which may be derived in the more standard way by equating voltage drops across the capacitor ( $V = Q/C$ ) and the inductor ( $V = -LdI/dt = -Ld^2Q/dt^2$ ). Note, however, that the Lagrangian approach has allowed us to forget about the correct choice of sign of the latter voltage, i.e. about complying with the Lenz rule. For more complex electric circuits this advantage may be very significant.

Moreover, the Lagrangian approach is extremely productive in relativistic physics, for example at the description of electromagnetic field interaction with charged relativistic particles – see, e.g., EM Sec. 10.9.

### 2.3. Hamiltonian

Does the fact that the Lagrange equation (19) has been derived using Eq. (17) mean that it always conserves energy? Not quite, because the “potential energy” participating in the latter equation, can depend not only on the generalized coordinates, but time as well. Let us start an analysis of this issue with the introduction of two new (and very important) notions: the *generalized momenta*

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} \quad (2.31)$$

<sup>8</sup> This attribution is not unique. Instead, one can use as the generalized coordinate the magnetic flux  $\Phi$  through inductive coil, related to the common voltage  $V$  across the circuit (Fig. 3) by the Faraday’s induction law  $V = -d\Phi/dt$ . With this choice,  $(-V)$  becomes the generalized velocity,  $E_L = \Phi^2/2L$ , the *potential* energy, and  $E_C = CV^2/2$ , the *kinetic* energy. It is straightforward to verify that for this choice, the Lagrange equations yield an equation of motion, which is equivalent to Eq. (29).

and the *Hamiltonian function*

$$H \equiv \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \equiv \sum_j p_j \dot{q}_j - L. \quad (2.32)$$

In order to see whether the Hamiltonian is conserved, let us differentiate its definition (32) over time:

$$\frac{dH}{dt} = \sum_j \left[ \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \dot{q}_j + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j \right] - \frac{dL}{dt}. \quad (2.33)$$

If we want to make use Eq. (19), the last derivative has to be calculated considering  $L$  as a function of independent arguments  $q_j$ ,  $\dot{q}_j$ , and  $t$ :

$$\frac{dL}{dt} = \sum_j \left( \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \dot{q}_j + \frac{\partial L}{\partial q_j} \frac{d}{dt} q_j \right) + \frac{\partial L}{\partial t}, \quad (2.34)$$

where the last term is the derivative of  $L$  as an *explicit* function of time. We see that the first term in the RHP of Eq. (33) cancels with the first term of Eq. (34). Moreover, using the Lagrange equation (19) for the second term in Eq. (33), we see that it cancels with the second term of Eq. (34). Thus we arrive at a very simple and important result:

$$\frac{dH}{dt} = - \frac{\partial L}{\partial t}. \quad (2.35)$$

The most important corollary of this equation is that if the Lagrangian function does not depend on time explicitly ( $\partial L / \partial t = 0$ ), the Hamiltonian an integral of motion:

$$H = \text{const.} \quad (2.36)$$

Let us see how it works on two first examples discussed in the previous section. For the example of a 1D particle, definition (31) of the generalized momentum yields

$$p_x \equiv \frac{\partial L}{\partial v} = mv, \quad (2.37)$$

so that it coincides with the usual momentum (or rather its  $x$ -component). According to Eq. (32), the Hamiltonian function for this case is

$$H \equiv p\dot{x} - L = m\dot{x}^2 - \left( \frac{m}{2} \dot{x}^2 - U \right) = \frac{m}{2} \dot{x}^2 + U, \quad (2.38)$$

and coincides with particle's mechanical energy  $E = T + U$ . Since the Lagrangian does not depend on time explicitly, both  $H$  and  $E$  are conserved.

However, it is not always that simple! Indeed, let us consider again the bead-on-the-rotating-ring problem (Fig. 1.6). In this case,

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta}, \quad (2.39)$$

and Eq. (32) yields:

$$\begin{aligned}
H \equiv p_\theta \dot{\theta} - L &= mR^2 \dot{\theta}^2 - \left[ \frac{m}{2} R^2 (\dot{\theta}^2 + \omega^2 \sin^2 \theta) + mgR \cos \theta \right] + \text{const} \\
&= \frac{m}{2} R^2 (\dot{\theta}^2 - \omega^2 \sin^2 \theta) - mgR \cos \theta + \text{const}.
\end{aligned} \tag{2.40}$$

This means that (as soon as  $\omega \neq 0$ ), the Hamiltonian function *differs* from the mechanical energy

$$E \equiv T + U = \frac{m}{2} R^2 (\dot{\theta}^2 + \omega^2 \sin^2 \theta) - mgR \cos \theta + \text{const}. \tag{2.41}$$

The difference,  $E - H = mR^2 \omega^2 \sin^2 \theta$  (besides an inconsequential constant), evidently changes at the bead motion along the ring, so that although  $H$  is an integral of motion (since  $\partial L / \partial t = 0$ ), energy  $E$  is *not* conserved.

Thus, Eq. (36) expresses a new conservation law, generally different of that of energy conservation. Let us examine when do these two laws coincide. In mathematics, there is a notion of a *homogeneous function*  $f(x_1, x_2, \dots)$  of degree  $\lambda$ , defined in the following way: for an arbitrary constant  $a$ ,

$$f(ax_1, ax_2, \dots) = a^\lambda f(x_1, x_2, \dots). \tag{2.42}$$

Such functions obey the following *Euler theorem*:<sup>9</sup>

$$\sum_j \frac{\partial f}{\partial x_j} x_j = \lambda f, \tag{2.43}$$

which may be readily proven by the differentiation of both parts of Eq. (42) over  $a$  and then setting this parameter to a particular value,  $a = 1$ . Now, consider the case when the kinetic energy is a quadratic form of all generalized velocities  $\dot{q}_j$ :

$$T = \sum_{j,j'} f_{jj'}(q_1, q_2, \dots, t) \dot{q}_j \dot{q}_{j'}, \tag{2.44}$$

with no other terms. It is evident that such  $T$  satisfies the definition of a homogeneous function of the velocities of degree two,<sup>10</sup> so that the Euler theorem (43) gives

$$\sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2K. \tag{2.45}$$

But since  $\partial L / \partial \dot{q}_j = \partial T / \partial \dot{q}_j$ , the left-hand part of Eq. (41) is exactly the first term in definition (32) of the Hamiltonian function, so that in this case

$$H = 2T - L = 2T - (K - U) = T + U = E. \tag{2.46}$$

Thus for the kinetic energy of the type (44), for example a free particle with the kinetic energy considered as a function of its Cartesian velocities,

<sup>9</sup> This is just one of many theorems bearing the name of the math giant Leonhard Euler (1707-1783).

<sup>10</sup> Such functions are called *quadratic-homogeneous*.

$$K = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) \quad (2.47)$$

the notions of the Hamiltonian function and mechanical energy are identical. (Indeed, some textbooks, very regrettably, do not distinguish these notions at all!) However, as we have seen from our bead-on-the-rotating-ring example, this is not always true. For that problem, the kinetic energy, in addition to the term proportional to  $\dot{\theta}^2$ , has another, velocity-independent term – see Eq. (23) – and hence is *not* a homogeneous function of the angular velocity.

#### 2.4. Other conservation laws

Looking at the Lagrange equation (19), we immediately see that if  $L \equiv T - U$  as a whole is independent of some generalized coordinate,<sup>11</sup>  $\partial L/\partial q_j = 0$ , the corresponding generalized momentum is an integral of motion:

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \text{const.} \quad (2.48)$$

For example, for a 1D particle with Lagrangian (21), momentum  $p_x$  is conserved if the potential energy is constant (the  $x$ -component of force is zero).

As another example, let us consider a 2D motion of a particle in the field of central forces. If we use polar coordinates  $r$  and  $\varphi$  in the role of the generalized coordinates, the Lagrangian function,<sup>12</sup>

$$L \equiv T - U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r), \quad (2.49)$$

is independent of  $\varphi$  and hence the corresponding generalized momentum,

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}, \quad (2.50)$$

is conserved. This is a particular case of the angular momentum conservation – see Eq. (1.24). Indeed, for the 2D motion within the  $[x,y]$  plane, the angular momentum vector,

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix}, \quad (2.51)$$

has only one nonvanishing component, perpendicular to the motion plane:

<sup>11</sup> Such coordinates are frequently called *cyclic*, because in some cases (like in the second example considered below) they represent periodic coordinates such as angles. However, this terminology may be misleading, because some “cyclic” coordinates (e.g.,  $x$  in the first example) have nothing to do with rotation.

<sup>12</sup> Note that here (and in Sec. 3.5 below),  $\dot{r}^2$  is just the square of the scalar derivative  $\dot{r}$ , rather than the square of vector  $\dot{\mathbf{r}}$ .

$$L_z = x(m\dot{y}) - y(m\dot{x}). \quad (2.52)$$

Differentiating the well-known relations between the polar and Cartesian coordinates (cf. MA Eq. (10.1)),

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad (2.53)$$

over time, and plugging the result into Eq. (52), we see that  $L_z = mr^2 \dot{\varphi} = p_\varphi$ .

Thus the Lagrangian formalism provide a powerful way of searching for non-evident integrals of motion. On the other hand, if such conserved quantity is evident or known *a priori*, it is helpful for the selection of the most appropriate generalized coordinates, giving the simplest Lagrange equations. For example, in the last problem, if we already knew that  $p_\varphi$  is conserved, this could provide a reason for including the corresponding coordinate, angle  $\varphi$ , into the list of the used generalized coordinates.