

Chapter 3. 1D Problems

In this chapter, we consider a few simple but important problems of particle motion constrained to move in one dimension, and effectively 1D systems including the famous Kepler problem of two particles interacting via the central potential.

3.1. One-dimensional and 1D-reducible systems

If a particle is confined to motion along a straight line (say, axis x), its position, of course, is completely defined by this coordinate. In this case, as we already know, the particle Lagrangian is given by Eq. (2.21):

$$L = L(x, \dot{x}, t) = \frac{m}{2} \dot{x}^2 - U(x, t), \quad (3.1)$$

so that the Lagrangian equation of motion (2.22)

$$m\ddot{x} = -\frac{\partial U(x, t)}{\partial x} \quad (3.2)$$

is just the x -component of the 2nd Newton law.

However, there is a substantially broader class of *effectively 1D systems* whose position, due to holonomic constraints and/or conservation laws, is also fully determined by one generalized coordinate q , and whose Lagrangians may be presented in the form similar to Eq. (1):

$$L = T_{\text{ef}}(\dot{q}) - U_{\text{ef}}(q, t), \quad T_{\text{ef}} = \frac{m_{\text{ef}}}{2} \dot{q}^2, \quad (3.3)$$

where m_{ef} is some constant which may be considered as the *effective mass* of the system, and the function U_{ef} its *effective potential energy*. In this case the Lagrangian equation (2.19) describing the system dynamics has the form

$$m_{\text{ef}} \ddot{q} = -\frac{\partial U_{\text{ef}}(q, t)}{\partial q}. \quad (3.4)$$

As an example, let us return again to our testbed system shown in Fig. 1.6. We have already seen that for that system, having one degree of freedom, the genuine kinetic energy T , expressed by the first of Eqs. (2.23), is *not* a quadratically-homogeneous function of the generalized velocity. However, the system's Lagrangian still may be presented in form (3):

$$L = \frac{m}{2} R^2 \dot{\theta}^2 + \frac{m}{2} R^2 \omega^2 \sin^2 \theta + mgR \cos \theta + \text{const} = T_{\text{ef}} - U_{\text{ef}}, \quad (3.5)$$

if we take

$$T_{\text{ef}} \equiv \frac{m}{2} R^2 \dot{\theta}^2, \quad U_{\text{ef}} \equiv -\frac{m}{2} R^2 \omega^2 \sin^2 \theta - mgR \cos \theta + \text{const}. \quad (3.6)$$

With this new partitioning of L , T_{ef} includes only a part of the full kinetic energy T of the bead, while U_{ef} includes not only the real potential energy U in the gravity field, but also a contribution due to the effective centrifugal “force”. (We will revisit this issue in the end of Chapter 5.)

Returning to the general case of effectively 1D systems with Lagrangian (3), let us calculate their Hamiltonian function, using its definition (2.32):

$$H = \frac{\partial L}{\partial \dot{q}} \dot{q} - L = m_{\text{ef}} \dot{q}^2 - (T_{\text{ef}} - U_{\text{ef}}) = T_{\text{ef}} + U_{\text{ef}}. \quad (3.7)$$

However, be careful! H may differ from the system’s mechanical energy E (which is defined as the sum $T + U$, not $T_{\text{ef}} + U_{\text{ef}}$). For example, for our testbed problem, $H \neq E$.

3.2. Equilibrium and stability

Autonomous systems are defined as dynamic systems with generalized forces Q_j which do not explicitly depend on time. For 1D (and effectively 1D) systems obeying Eq. (4), this means that their function U_{ef} does not depend on time explicitly. Such systems, at appropriate initial conditions, may stay in equilibrium at one or several *stationary* (also called “fixed”) *points* q_n , corresponding to either the minimum or a maximum of the effective potential energy (see Fig. 1):

$$\frac{dU_{\text{ef}}}{dq}(q_n) = 0. \quad (3.8)$$

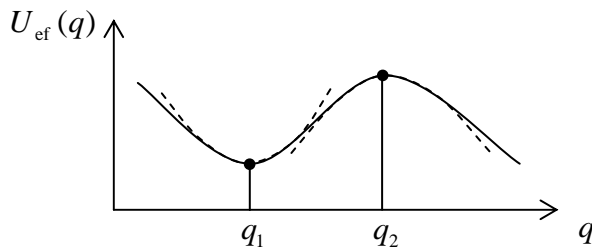


Fig. 3.1. Dashed lines show the quadratic approximations (10) of the potential energy profile near a stable (q_1) and unstable (q_2) fixed points – schematically.

In order to explore the *stability* of such fixed points, let us analyze dynamics of small deviations

$$\tilde{q}(t) \equiv q(t) - q_n \quad (3.9)$$

from the equilibrium. For that, let us expand function $U_{\text{ef}}(q)$ in the Taylor series (see, e.g., MA Eq. (4.3)) near the fixed point:

$$U_{\text{ef}}(q) = U_{\text{ef}}(q_n) + \frac{1}{2} \frac{d^2 U_{\text{ef}}}{dq^2}(q_n) \tilde{q}^2 + \dots, \quad (3.10)$$

where we have dropped the term linear in deviation \tilde{q} , due to Eq. (8). Constant $U_{\text{ef}}(q_n)$ is arbitrary and does not affect motion, and the higher terms are small for very small deviations,¹ so that stability is determined by the sign of the second derivative in the fixed point,

¹ These terms are only important in the very special cases when coefficient κ_{ef} equals zero *exactly*.

$$\kappa_{\text{ef}} \equiv \frac{d^2 U_{\text{ef}}}{dq^2}(q_n) \quad (3.11)$$

which plays the role of the effective spring constant.

Indeed, neglecting the higher terms of the Taylor expansion, we see that Eq. (4) takes the familiar form - cf. Eq. (1.1):

$$m_{\text{ef}} \ddot{\tilde{q}} + \kappa_{\text{ef}} \tilde{q} = 0. \quad (3.12)$$

From the mathematical standpoint, this is an ordinary, linear differential equation of the second order with constant coefficients. The theory of such equations tells us that their general solution may be presented as

$$\tilde{q}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad (3.13)$$

where constants $c_{1,2}$ are determined by the initial conditions, while the so-called “*characteristic exponents*” $\lambda_{1,2}$ are completely defined by the equation itself. In order to find the exponents, it is sufficient to plug just one partial solution, $\exp\{\lambda t\}$, into the equation. In our simple case (12), this yields the following *characteristic equation*:

$$m_{\text{ef}} \lambda^2 + \kappa_{\text{ef}} = 0. \quad (3.14)$$

It shows that fixed points may differ. If the ratio $\kappa_{\text{ef}}/m_{\text{ef}}$ is positive,² i.e. the fixed point corresponds to the minimum of potential energy (e.g., point q_1 in Fig. 1), the characteristic equation yields

$$\lambda_{1,2} = \pm i\omega_0, \quad \omega_0 \equiv \left(\frac{\kappa_{\text{ef}}}{m_{\text{ef}}} \right)^{1/2}, \quad (3.15)$$

(where i is the imaginary unity, $i^2 = -1$), so that Eq. (13) describes sinusoidal oscillations of the system,

$$\tilde{q}(t) = c_1 e^{+i\omega_0 t} + c_2 e^{-i\omega_0 t} = c_c \cos \omega_0 t + c_s \sin \omega_0 t, \quad (3.16)$$

with “*eigenfrequency*” ω_0 , about the fixed point which is hereby *stable*. On the other hand, at the potential energy maximum ($\kappa_{\text{ef}} < 0$, see point q_2 in Fig. 1), we get

$$\lambda_{1,2} = \pm \lambda, \quad \lambda \equiv \left(\frac{|\kappa_{\text{ef}}|}{m_{\text{ef}}} \right)^{1/2}, \quad \tilde{q}(t) = c_1 e^{+\lambda t} + c_2 e^{-\lambda t}. \quad (3.17)$$

Since the solution has an exponentially growing part,³ the fixed point is *unstable*.

The same result may be obtained by another way which may be readily generalized to more complex systems (e.g., those with dissipation). Let us Note that the *quadratic* expansion of function $U_{\text{ef}}(q)$, given by Eq. (8), is equivalent to a *linear* expansion of the effective force:

² In what follows, I will assume that the effective mass m_{ef} is positive, which is true in most (but not all!) dynamic systems, because the generalization is straightforward.

³ Mathematically, the growing part vanishes at some special (exact) initial conditions which give $c_1 = 0$. However, the futility of this argument for real physical systems should be obvious.

$$-\left. \frac{dU_{\text{ef}}}{dq} \right|_{q \approx q_n} \approx -\kappa_{\text{ef}} \tilde{q}, \quad (3.18)$$

immediately resulting in the linear equation (12). Hence, in order to analyze stability of a fixed point q_n , it is sufficient to “linearize” the equation of motion in small deviations, and study possible solutions to the resulting linear equation.

As an example, let us return to our testbed problem (Fig. 1.6) whose function U_{ef} we already know – see Eq. (6). From it, the equation of motion (4) becomes

$$mR^2 \ddot{\theta} = -\frac{dU_{\text{ef}}}{d\theta} = mR^2 [\omega^2 \cos \theta - \Omega^2] \sin \theta, \quad (3.19)$$

where Ω is the frequency of small oscillations of the system at $\omega = 0$ - see Eq. (2.26). From requirement (8), we see that the system may have four fixed points:

$$\theta_1 = 0, \quad \theta_2 = \pi, \quad \theta_{3,4} = \pm \arccos \frac{\Omega^2}{\omega^2}, \quad (3.20)$$

The last two fixed points, corresponding to the bead rotating on either side of the ring, exist only if the angular speed ω of ring rotation exceeds Ω . (If $\omega \gg \Omega$, then $\theta_{3,4} \rightarrow \pm \pi/2$, i.e. the stationary positions approach the horizontal diameter of the ring, in accordance with common sense.)

In order to analyze the fixed point stability, similarly to Eq. (9), we plug $\theta = \theta_n + \tilde{\theta}$ into Eq. (19) and Taylor-expand the trigonometric functions of θ up to the first term in $\tilde{\theta}$:

$$\ddot{\tilde{\theta}} = [\omega^2 (\cos \theta_n - \sin \theta_n \tilde{\theta}) - \Omega^2] (\sin \theta_n + \cos \theta_n \tilde{\theta}). \quad (3.21)$$

For the first fixed point (corresponding to the bead position at the bottom of the ring), $\theta_1 = 0$, we have $\cos \theta_1 = 1$ and $\sin \theta_1 = 0$, so that Eq. (21) is reduced to

$$\ddot{\tilde{\theta}} = (\Omega^2 - \omega^2) \tilde{\theta}, \quad (3.22)$$

so that the characteristic equation yields

$$\lambda^2 = \Omega^2 - \omega^2, \quad \text{for } \theta \approx \theta_1. \quad (3.23a)$$

This result shows that this fixed point is stable if $\omega < \Omega$, and unstable beyond this threshold, i.e. as soon as fixed points $\theta_{3,4}$ exist. Absolutely similar calculations for other fixed points yield

$$\lambda^2 = \Omega^2 + \omega^2, \quad \text{for } \theta \approx \theta_2, \quad (3.23b)$$

$$\lambda^2 = \omega^2 - \Omega^2, \quad \text{for } \theta \approx \theta_{3,4}. \quad (3.23c)$$

These results show that the second fixed point (bead on the top of the ring), In correspondence with common sense, is always unstable, while the third and fourth fixed points are always stable when they exist (at $\omega > \Omega$).

Thus, our fixed-point analysis may be summarized in a simple way: an increase of the ring rotation speed ω beyond a certain threshold value, equal to Ω (2.26), causes the bead to move on one of

the ring sides, oscillating about one of the fixed points $\theta_{3,4}$.⁴ Together with rotation about the vertical axis, this may yield quite a complex trajectory, if observed from a lab frame!

Later in this course, we will repeatedly use the linearization of the equations of motion for the analysis of stability of more complex systems, including those with energy dissipation

3.3. Hamiltonian 1D systems

Autonomous systems described by time-independent Lagrangians, are frequently called *Hamiltonian*, because their Hamiltonian function H (again, not necessarily equal to the genuine mechanical energy E !) is conserved. In our current 1D case, described by Eq. (3),

$$H = \frac{m_{\text{ef}}}{2} \dot{q}^2 + U_{\text{ef}}(q) = \text{const}. \quad (3.24)$$

This is the first integral motion. Solving it for \dot{q} , we get the first-order differential equation

$$\frac{dq}{dt} = \pm \left\{ \frac{2}{m_{\text{ef}}} [H - U_{\text{ef}}(q)] \right\}^{1/2} \quad (3.25)$$

which may be further integrated as

$$\pm \left(\frac{m_{\text{ef}}}{2} \right)^{1/2} \int_{q(t_0)}^{q(t)} \frac{dq'}{[H - U_{\text{ef}}(q')]^{1/2}} = t - t_0. \quad (3.26)$$

This result gives the reciprocal form, $t(q)$, of the desired law of system motion, $q(t)$. (The two constants, H and t_0 , participating in this result, as well as the proper choice of the sign before the integral, are determined by initial conditions – see below.) Of course, integral (26) still has to be worked out, either analytically or numerically, but even the latter procedure is typically much easier than the numerical integration of the initial, second-order differential equation of motion – see, e.g., MA Eqs. (5.2) and (5.3).

Moreover, Eqs. (24)-(25) also allow a general classification of 1D system motion. Indeed:

(i) If $H > U_{\text{ef}}(x)$ in the whole range of interest, the effective kinetic energy T_{ef} (3) is always positive. Hence the motion velocity dq/dt cannot change sign, and retains the sign it had initially. This is the unbound motion in one direction (Fig. 2a).

(ii) Now let the particle approach a “*classical turning point*” A where $H = U_{\text{ef}}(x)$ - see Fig. 2b.⁵ According to Eqs. (25), (26), at that point the particle velocity vanishes, while its acceleration, according to Eq. (4), is still finite. Evidently, this corresponds to the particle reflection from the “potential wall”, with the change of velocity sign.

⁴ Note that fixed points $\theta_{3,4}$ correspond to two minima of the *effective* potential energy (6) of the bead rather than its genuine potential energy $U = -mgR \cos \theta$ (whose minimum is located at point $\theta_1 = 0$, for any ω).

⁵ This terminology comes from quantum mechanics which shows that actually a particle (or rather its wavefunction) can, to a certain extent, penetrate the “classically forbidden range” where $H < U_{\text{ef}}(x)$. However, for macroscopic bodies, the depth of penetration is extremely small – see, e.g., QM Eqs. (2.40), (2.73) and their discussion, so for them the classical analysis is pretty valid.

(iii) If, after the reflection from point A , the particle runs into another classical turning point B (Fig. 2c), the reflection process is repeated again and again, so that the particle is bound to a periodic motion between two turning points.

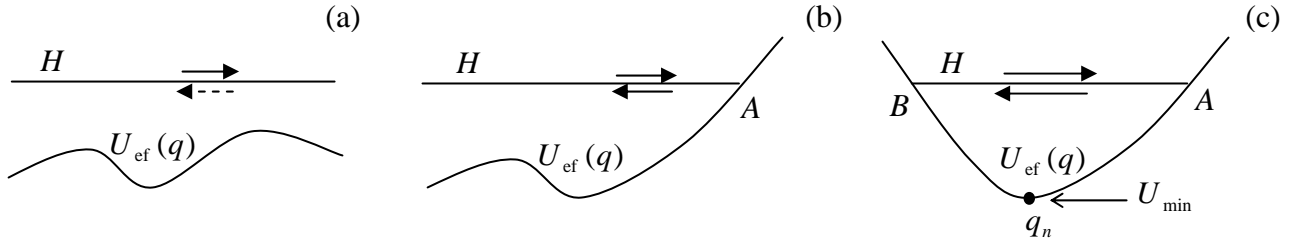


Fig. 3.2. Energy diagram for three different cases of the effectively 1D motion: (a) unbound motion, with the initial velocity sign conserved, (b) reflection from the “classical turning point”, accompanied with the velocity sign change, and (c) bound, periodic motion between two classical turning points.

The last case of *periodic oscillations* presents large practical interest, and the whole next chapter will be devoted to a detailed analysis of this phenomenon and numerous associated effects. Here I will only note that Eq. (26) immediately enables us to calculate the oscillation period:⁶

$$T = 2 \left(\frac{m_{\text{ef}}}{2} \right)^{1/2} \int_B^A \frac{dq}{[H - U_{\text{ef}}(q)]^{1/2}}, \quad (3.27)$$

where the additional factor 2 accounts for two time intervals: for the path from B to A and back (Fig. 2c). Indeed, according to Eq. (25), in each classically allowed point q the velocity magnitude is the same, so that these intervals are equal to each other.

Let us link Eq. (27) to the fixed point analysis carried out in the previous section. As Fig. 2c shows, if H is reduced to approach U_{min} , the oscillations described by Eq. (27) take place at the very bottom of “potential well”, about a stable fixed point q_n . Hence, if the potential energy profile is smooth enough, we can limit the Taylor expansion (10) by the quadratic term. Plugging it into Eq. (27), and using the mirror symmetry of this particular problem about fixed point q_n , we get

$$T = 4 \left(\frac{m_{\text{ef}}}{2} \right)^{1/2} \int_0^A \frac{d\tilde{q}}{[H - (U_{\text{min}} + \kappa_{\text{ef}} \tilde{q}^2 / 2)]^{1/2}} = \frac{4}{\omega_0} I, \quad I \equiv \int_0^1 \frac{d\xi}{(1 - \xi^2)^{1/2}}, \quad (3.28)$$

where $A \equiv (2/\kappa_{\text{ef}})^{1/2}[H - U_{\text{min}}]^{1/2}$ is the classical turning point, i.e. the oscillation amplitude, and ω_0 is the eigenfrequency given by Eq. (15). Taking into account that the elementary integral I in that equation equals $\pi/2$,⁷ we finally get

$$T = \frac{2\pi}{\omega_0}, \quad (3.29)$$

⁶ Unfortunately, T is the traditional notation for the oscillation period, as well as the kinetic energy. Let me hope this would not lead to confusion, because the difference between these two usages is clear from the context.

⁷ Introducing a new variable ζ as $\xi \equiv \sin \zeta$, we have $d\xi = \cos \zeta d\zeta = (1 - \xi^2)^{1/2} d\zeta$, so that $I = \int_0^{\pi/2} d\zeta = \frac{\pi}{2}$.

as it should be. Notice that the frequency does not depend on the oscillation amplitude a , i.e. on the difference $(H - U_{\min})$ - while it is small.

3.4. Planetary problems

Leaving a more detailed study of oscillations for the next chapter, let us now discuss the so-called “planetary” two-particle systems⁸ whose description, somewhat surprisingly, may be also reduced to an effectively 1D problem.

Consider two particles which interact via a conservative, central force $\mathbf{F}_{12} = -\mathbf{F}_{21} = \mathbf{n}_r F(r)$, where r and \mathbf{n}_r are, respectively, the magnitude and direction of the “distance vector” $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ connecting the two particles (Fig. 3). The most evident examples of such interaction include the electrostatic (“Coulomb”) interaction of charged particles, and the Newtonian gravity. Generally, two particles moving without constrains in 3D space, have $3 + 3 = 6$ degrees of freedom which may be described, e.g., by their Cartesian coordinates $\{x_1, y_1, z_1, x_2, y_2, z_2\}$. However, the following three tricks allow the number of essential degrees of freedom to be reduced to just one.

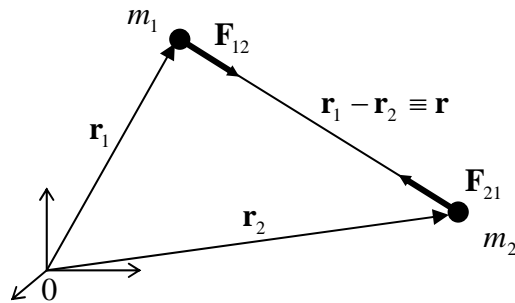


Fig. 3.3. The general “planetary” system.

First, the central, conservative force of the particle interaction may be described by time-independent potential energy $U(r)$. Hence the Lagrangian of the system is

$$L = \frac{m_1}{2} \dot{\mathbf{r}}_1^2 + \frac{m_2}{2} \dot{\mathbf{r}}_2^2 - U(r). \quad (3.30)$$

Let us transfer to new six generalized coordinates: 3 components of the distance vector

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2, \quad (3.31)$$

and 3 components of vector

$$\mathbf{R} \equiv \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M}, \quad M \equiv m_1 + m_2, \quad (3.32)$$

corresponding to the *center of mass* of the system. Solving the system of 2 linear equations (31) and (32) for the initial radius-vectors \mathbf{r}_1 and \mathbf{r}_2 , we get

⁸ This name is of course very conditional, because this group of problems includes, for example, charged particle scattering (see Sec. 3.7 below).

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M} \mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M} \mathbf{r}. \quad (3.33)$$

Plugging these relations into Eq. (30), we may rewrite the Lagrangian as

$$L = \frac{M}{2} \dot{\mathbf{R}}^2 + \frac{m}{2} \dot{\mathbf{r}}^2 - U(r), \quad (3.34)$$

where m is the so-called “reduced” mass:

$$m \equiv \frac{m_1 m_2}{M}; \quad \text{or} \quad \frac{1}{m} \equiv \frac{1}{m_1} + \frac{1}{m_2}. \quad (3.35)$$

Since the Lagrangian depends only on $\dot{\mathbf{R}}$ rather than \mathbf{R} itself, according to our discussion in Sec. 2.4, all three generalized momenta, corresponding to the Cartesian components of \mathbf{R} , are conserved:

$$P_j \equiv \frac{\partial L}{\partial \dot{R}_j} = M \dot{R}_j = \text{const}, \quad j = 1, 2, 3. \quad (3.36)$$

Physically, this is just the conservation law for the full momentum $\mathbf{P} \equiv M\mathbf{R}$ of our system, due to absence of external forces. We have actually postulated this law – see Eq. (1.10) – but now can attribute it to a certain geometric point – the center of mass. In particular, since the center moves with constant velocity in the (unspecified) inertial reference frame we were using to write Eq. (30), we can create a new inertial frame with the origin at that point. In this new frame, $\mathbf{R} = 0$, while \mathbf{r} remains the same (because the frame transfer vector adds equally to \mathbf{r}_1 and \mathbf{r}_2 , and cancels in $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$), so that the Lagrangian (34) is now reduced to

$$L = \frac{m}{2} \dot{\mathbf{r}}^2 - U(r). \quad (3.37)$$

Thus our problem has been reduced to just 3 degrees of freedom – the components of vector \mathbf{r} .⁹ Moreover, Eq. (37) shows that dynamics of vector \mathbf{r} in our initial, two-particle system is identical to that of the radius-vector of a *single particle* with the effective mass m , moving in the central potential field $U(r)$.

3.5. 2nd Kepler law

One more degree of freedom may be excluded from the planetary problem by noticing that according to Eq. (1.24), the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of our effective particle is also conserved, both in magnitude and direction. Since the direction of \mathbf{L} is, by its definition, perpendicular to both of \mathbf{r} and $\mathbf{v} = \mathbf{p}/m$, this means that the particle motion is confined to a plane (whose orientation in space is determined by the initial directions of \mathbf{r} and \mathbf{v}). Hence we can completely describe the particle position by just two coordinates in that plane, for example by distance r to the center, and the polar angle φ . In these coordinates, Eq. (37) takes the form identical to Eq. (2.49):

⁹ Note also that according to the second form of Eq. (27), the reduced mass m is lower than that of the lightest component of the 2-body system. If one of $m_{1,2}$ is much less than its counterpart (like it is in most star-planet or planet-satellite systems), then with good precision, $m \cong \min [m_1, m_2]$, so that \mathbf{r} is just the radius-vector of the lightest component moving in a fixed central field $U(r)$.

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r). \quad (3.38)$$

As if the reduction from six to just two degrees of freedom, described above, was not fascinating enough, it can be continued to exclude one more coordinate, angle φ . Indeed, so far we have used the conservation of the angular momentum *direction*. Now let us use the conservation of its *magnitude*, in form (2.50):¹⁰

$$L_z = mr^2\dot{\varphi} = \text{const.} \quad (3.39)$$

A direct corollary of this conservation is the so-called *2nd Kepler law*:¹¹ the radius-vector \mathbf{r} sweeps equal areas A in equal times. Indeed, in the linear approximation in $dA \ll A$, the area differential dA equals to that of a narrow right triangle with a base being the arc differential length $rd\varphi$, and height equal to r - see Fig. 4. As a result, according to Eq. (39), the time derivative of the area,

$$\frac{dA}{dt} = \frac{r(rd\varphi)/2}{dt} = \frac{1}{2}r^2\dot{\varphi} = \frac{L_z}{2m}, \quad (3.40)$$

remains constant. Integration of this equation over an arbitrary (not necessarily small!) time interval proves the *2nd Kepler law*.

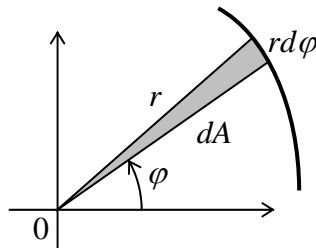


Fig. 3.4. Area differential in the polar coordinates.

Notice now that since $\partial L / \partial t = 0$, the Hamiltonian function H is also conserved, and since, according to Eq. (38), the kinetic energy of the system is a quadratic-homogeneous function of the generalized velocities \dot{r} and $\dot{\varphi}$, $H = E$, so that the system energy E ,

$$E = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\varphi}^2 + U(r), \quad (3.41)$$

is also a first integral of motion.¹² But according to Eq. (31), the second term of Eq. (33) may be presented as

¹⁰ Here index z stands for the coordinate perpendicular to the motion plane. Since other components of the angular momentum are equal zero, the index is not really necessary, but I will still use it, just to make a clear distinction between the angular momentum and the Lagrangian function.

¹¹ One of 3 laws deduced almost exactly 400 years ago by Johannes Kepler (1571 – 1630), from the extremely detailed astronomical data collected by Tycho Brahe (1546-1601). In turn, the set of three Kepler laws were the main basis for Isaac Newton's discovery of the gravity law. This is how physics marched...

¹² One may claim that this was evident from the very beginning, because the effective particle of mass m moves in a potential field $U(r)$ which conserves energy.

$$\frac{m}{2} r^2 \dot{\varphi}^2 = \frac{L_z^2}{2mr^2}, \quad (3.42)$$

so that energy (41) may be presented as that of a particle moving along one axis r ,

$$E = \frac{m}{2} \dot{r}^2 + U_{\text{ef}}(r), \quad (3.43)$$

in an effective one-dimensional potential

$$U_{\text{ef}}(r) \equiv U(r) + \frac{L_z^2}{2mr^2}. \quad (3.44)$$

So we have reduced the planetary motion to an effectively 1D system.¹³

Now we can proceed like in Sec. 3, with due respect for the very specific effective potential (44) which, in particular, diverges at $r \rightarrow 0$ (besides the very special case of a purely radial motion, $L_z = 0$). In particular, we can solve Eq. (43) for dr/dt to write, similarly to Eq.

$$dt = \left(\frac{m}{2}\right)^{1/2} \frac{dr}{[E - U_{\text{ef}}(r)]^{1/2}}. \quad (3.45)$$

Integration of this relation allows us not only to calculate the direct relation between time t and distance r - cf. Eq. (26):

$$t = \pm \left(\frac{m}{2}\right)^{1/2} \int \frac{dr}{[E - U_{\text{ef}}(r)]^{1/2}} = \pm \left(\frac{m}{2}\right)^{1/2} \int \frac{dr}{[E - U(r) - L_z^2 / 2mr^2]^{1/2}}, \quad (3.46)$$

but also proceed, using Eq. (31), to the calculation of angle φ :

$$\varphi = \int \dot{\varphi} dt = \frac{L_z}{m} \int \frac{dt}{r^2}. \quad (3.47)$$

Plugging dt from Eq. (45) into Eq. (47), we get an explicit expression for particle's trajectory $\varphi(r)$:

$$\varphi = \pm \frac{L_z}{(2m)^{1/2}} \int \frac{dr}{r^2 [E - U_{\text{ef}}(r)]^{1/2}} = \pm \frac{L_z}{(2m)^{1/2}} \int \frac{dr}{r^2 [E - U(r) - L_z^2 / 2mr^2]^{1/2}}. \quad (3.48)$$

Note that according to Eq. (39), derivative $d\varphi/dt$ does *not* change sign at the reflection from any classical turning point $r \neq 0$, so that, in contrast to Eq. (38), the sign in Eq. (40) is determined by the initial conditions and cannot change during the motion, unless the particle runs exactly into the origin.

Let us use these general results, first of all, for the classification of the planetary motion. The following cases should be distinguished.

If the particle interaction is *attractive*, with $U(r)$ either converging or diverging slower than $1/r^2$ at $r \rightarrow 0$,¹⁴ the energy profile $U_{\text{ef}}(r)$ has the shape shown schematically in Fig. 5. (Following tradition, in

¹³ Note that it has been done in a way different than for our “testbed” problem (Fig. 1.6) – see Problem 6. In order to emphasize this difference, I will write E instead of H , though for the planetary problem we are discussing now these two notions coincide.

¹⁴ If the divergence is stronger than $1/r^2$, particle may drop on the center even if $L_z \neq 0$ – see Problem 5.

what follows I will select the arbitrary constant in the way to provide $U_{\text{ef}} \rightarrow 0$ at $r \rightarrow \infty$.) This is true, in particular, for the very important case

$$U(r) = -\frac{\alpha}{r}, \quad \alpha > 0, \quad (3.49)$$

which describes, in particular, the Coulomb interaction of two particles with electric charges of the opposite sign, and Newton's gravity law.¹⁵ The angular-momentum term dominates at small distances r , providing an effective potential with a minimum at some distance r_0 . According to our analysis in Sec. 3 (see Fig. 2), this potential profile may sustain two types of 1D motion, depending on the energy E (which is of course determined by the initial conditions):

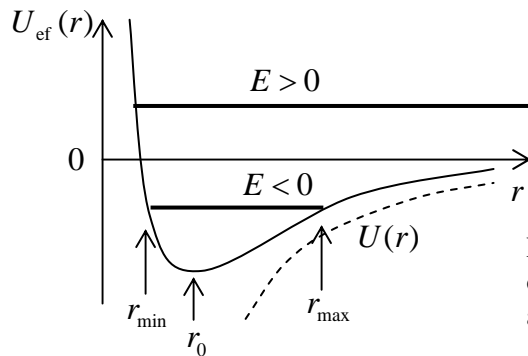


Fig. 3.5. The effective potential profiles of, and two types of motion in an attractive central field.

(i) If $E > 0$, there is only one classical turning point where $E = U_{\text{ef}}$, so that distance r either grows with time from the very beginning, or (if the initial value of \dot{r} was negative) first decreases and then, after the reflection from the increasing U_{ef} , starts to grow indefinitely. The latter case, of course, describe *scattering*.

(ii) On the opposite, if the energy is within the range

$$U_{\text{ef}}(r_0) \leq E < 0, \quad (3.50)$$

the system moves periodically between two classical turning points r_{min} and r_{max} . These oscillations of distance r correspond to the bound orbital motion of our effective particle about the attracting center.

In the opposite case when the interaction is *repulsive*, $U(r) > 0$, the addition of the positive angular energy term only increases the trend, and only the scattering situation is possible.

Let us start with the discussion of the bound motion, with energy within the range (50). If energy has its minimal possible value,

$$E = U_{\text{ef}}(r_0) = \min[U_{\text{ef}}(r)], \quad (3.51)$$

the distance cannot change, $r = r_0 = \text{const}$, so that the orbit is circular, with the radius r_0 satisfying the condition $dU_{\text{ef}}/dr = 0$. Let us see whether this result have an elementary explanation. Using Eq. (44) we see that the condition for r_0 may be written as

¹⁵ In the latter case, $\alpha = Gm_1m_2 = GMm$, where G in Newton's gravity constant.

$$\frac{L_z^2}{mr_0^3} = \frac{dU}{dr} \Big|_{r=r_0}. \quad (3.52)$$

Since in a circular motion, velocity \mathbf{v} is perpendicular to the radius vector \mathbf{r} , L_z is just mr_0v , the left-hand part of Eq. (52) equals mv^2/r_0 , while its right-hand part is just the magnitude of the attractive force, so that this equation expresses just the well-known 2nd Newton law for the circular motion. Plugging this result into Eq. (39) and integrating it, we get a linear law of angle change, $\varphi = \omega t + \text{const}$, with angular velocity

$$\omega = \frac{L_z}{mr_0^2}. \quad (3.53)$$

and hence the rotation period

$$T_\varphi = \frac{2\pi}{\omega} = \frac{2\pi r_0}{v}. \quad (3.54)$$

Now, let the energy be slightly above its minimum value. This should not affect result (54) too much. However, repeating the analysis of Sec. 3 (see, in particular, Eq. (27)), we see that distance r now oscillates with period

$$T_r = (2m)^{1/2} \int_{r_{\min}}^{r_{\max}} \frac{dr}{[E - U(r) - L_z^2 / 2mr^2]^{1/2}}. \quad (3.55)$$

It is clear that T_r has, in general, nothing common with T_φ , and that this difference may persist for any value of $E < 0$. This means that the change of angle φ between two sequential points of the nearest approach, which follows from Eq. (48),

$$|\Delta\varphi| = 2 \frac{L_z}{(2m)^{1/2}} \int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2 [E - U(r) - L_z^2 / 2mr^2]^{1/2}}, \quad (3.56)$$

is generally different from 2π . Hence, the general trajectory of the bound motion has a spiral shape – see, e.g., Fig. 6.

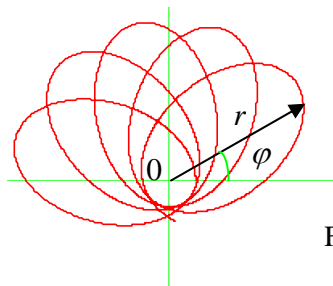


Fig. 3.6. Typical orbit in a non-Coulomb central field.

3.6. 1st and 3rd Kepler laws

There is however, a very important particular case, namely that of the “Coulomb” potential described by Eq. (49). Plugging this potential into Eq. (48), we get

$$\varphi = \pm \frac{L_z}{(2m)^{1/2}} \int \frac{dr}{r^2 [E + \alpha/r - L_z^2/2mr^2]^{1/2}}. \quad (3.57)$$

This is a table integral (see, e.g., MA Eq. (6.3)), equal to

$$\varphi = \pm \arccos \frac{L_z^2/m\alpha r - 1}{(1 + 2EL_z^2/m\alpha^2)^{1/2}} + \text{const}. \quad (3.58)$$

At $E < 0$, the orbit a closed line, because we may change φ by any integer number of 2π increments without changing r . In order to explore this trajectory further, it is convenient to introduce the following parameter combinations:

$$p \equiv \frac{L_z^2}{m\alpha}, \quad e \equiv \left[1 + \frac{2EL_z^2}{m\alpha^2} \right]^{1/2}. \quad (3.59)$$

Their physical meaning is very simple. Indeed, according to the general Eq. (52), in the Coulomb potential, for which $dU/dr = \alpha/r^2$, the circular orbit radius $r_0 = L_z^2/m\alpha \equiv p$,¹⁶ and

$$\min[U_{\text{ef}}(r)] \equiv U_{\text{ef}}(r_0) = -\frac{\alpha^2 m}{2L_z^2}, \quad (3.60)$$

Using this equality, parameter e (called *eccentricity*) may be presented just as

$$e = \left\{ 1 - \frac{E}{\min[U_{\text{ef}}(r)]} \right\}^{1/2}. \quad (3.61)$$

With this notation, Eq. (58) reduces to the following simple form:

$$r = \frac{p}{1 + e \cos(\varphi + \text{const})}. \quad (3.62)$$

Analytical geometry tells us that Eq. (62), with $e < 1$, is one of canonical forms for presentation of an *ellipse*, with one of its two foci located at the origin.¹⁷ Figure 7 shows the relation between the main dimensions of the ellipse and parameters p and e .¹⁸ In particular, the major axis a and minor axis b are simply related to p and e and hence, via Eqs. (59), to the physical parameters of the motion:

$$a = \frac{p}{1 - e^2} = \frac{\alpha}{2|E|}, \quad b = \frac{p}{(1 - e^2)^{1/2}} = \frac{L_z}{(2m|E|)^{1/2}}. \quad (3.63)$$

¹⁶ Mathematicians prefer a more solemn terminology: parameter $2p$ is called the *latus rectum*.

¹⁷ This fact is known as the *1st Kepler law*.

¹⁸ In this figure, the constant in Eq. (60) is assumed to be zero. It is evident that a different choice of the constant corresponds just to a turn of the ellipse about the origin.

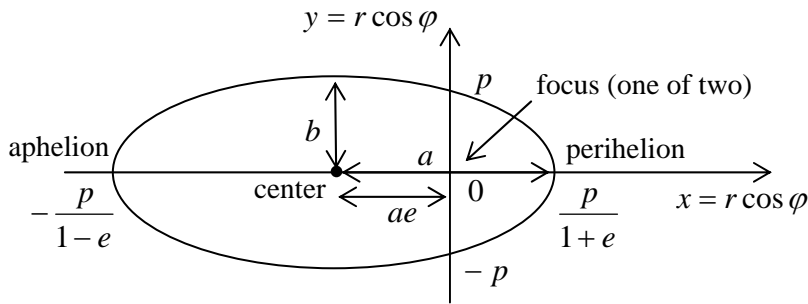


Fig. 3.7. Ellipse, and its special points and dimensions.

Equation (61) shows that if the initial conditions correspond to a nearly minimal energy, then $e \approx 0$, and the orbit is almost circular, with $r(\varphi) \cong r_0 = p$. On the contrary, as E is increased to approach zero (its maximum value for the closed orbit), then $e \rightarrow 1$, so that the aphelion point $r_{\max} = p/(1 - e)$ tends to infinity, i.e. the orbit becomes extremely extended. If the energy is exactly zero, Eq. (62) (with $e = 1$) is still valid for all values of φ (except for one special point $\varphi = \pi$ where r becomes infinite) and describes a *parabolic* (i.e. open) trajectory. At $E > 0$, Eq. (62) is still valid within a certain sector of angles φ (in which it yields positive results for r), and describes an open, *hyperbolic* trajectory (see the next section).

For $E < 0$, the above relations also allow a ready calculation of the rotation period $T = T_r = T_\varphi$. (In the case of a closed trajectory, T_r and T_φ have to coincide.) Indeed, it is well known that the ellipse area $A = \pi ab$. But according to the 2nd Kepler law (40), $dA/dt = L_z/2m = \text{const}$. Hence

$$T = \frac{A}{dA/dt} = \frac{\pi ab}{L_z/2m}. \quad (3.64a)$$

Using Eqs. (59) and (63), this result may be presented in several other forms:

$$T = \frac{\pi p^2}{(1-e^2)^{3/2} (L_z/2m)} = \pi \alpha \left(\frac{m}{2|E|^3} \right)^{1/2} = 2\pi a^{3/2} \left(\frac{m}{\alpha} \right)^{1/2}. \quad (3.64b)$$

Since for the Newtonian gravity $\alpha \propto m_1 m_2 = mM \propto m$, the last form of Eq. (64b) yields the 3rd Kepler law: periods of planetary motion (in the same central field, say that of the Sun) scale as $T^2 \propto a^3$. Notice that in contrast to the 2nd Kepler law (which is valid for any central field), the 1st and the 3rd Kepler laws are potential-specific.

3.7. Classical theory of elastic scattering

If $E > 0$, the motion is unbound for any interaction potential. In this case, the two most important parameters of the particle trajectory are the *scattering angle* θ and *impact parameter* b (Fig. 8), and the main task for theory is to find the relation between them in the given potential $U(r)$. For that, it is

convenient to Note that b is related to two conserved quantities, the particle energy¹⁹ E and angular momentum L_z , in a simple way:²⁰

$$L_z = b(2mE)^{1/2}. \quad (3.65)$$

Hence the angular contribution to the effective potential (44) may be presented as

$$\frac{L_z^2}{2mr^2} = E \frac{b^2}{r^2}. \quad (3.66)$$

Second, according to Eq. (48), the trajectory sections from infinity to the nearest approach point ($r = r_{\min}$), and from that point to infinity, have to be similar, and hence correspond to equal angle changes φ_0 (Fig. 8). Hence we can apply the general Eq. (48) to just one of the sections, say $[r_{\min}, \infty]$, to find the scattering angle:

$$\theta = \pi - 2\varphi_0 = \pi - 2 \frac{L_z}{(2m)^{1/2}} \int_{r_{\min}}^{\infty} \frac{dr}{r^2 [E - U(r) - L_z^2 / 2mr^2]^{1/2}} = \pi - 2 \int_{r_{\min}}^{\infty} \frac{bdr}{r^2 [1 - U(r)/E - b^2/r^2]^{1/2}}. \quad (3.67)$$

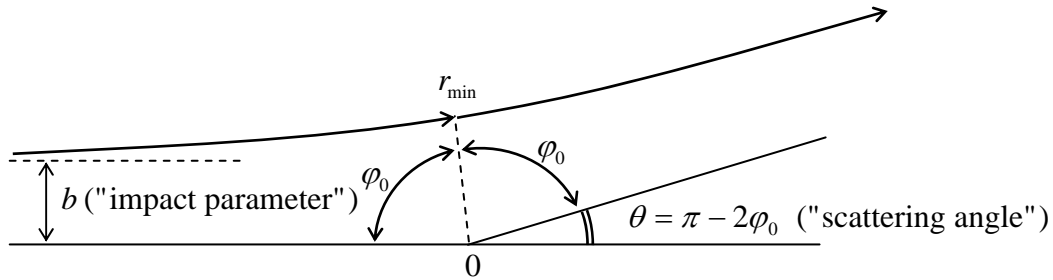


Fig. 3.8. Main geometric parameters of the scattering problem.

In particular, for the Coulomb potential (49), now with an arbitrary sign of α , we can apply the same table integral as in the previous section to get²¹

$$\theta = \left| \pi - 2 \arccos \frac{\alpha / 2Eb}{[1 + (\alpha / 2Eb)^2]^{1/2}} \right|. \quad (3.68a)$$

This result may be more conveniently rewritten as

$$\tan \frac{|\theta|}{2} = \frac{|\alpha|}{2Eb}. \quad (3.68b)$$

Very clearly, the scattering angle's magnitude increases with the potential strength α and decreases with both the particle energy and impact parameter.

¹⁹ The energy conservation law is frequently emphasized by calling this process the *elastic scattering*.

²⁰ Indeed, at $r \gg b$, the definition $\mathbf{L} = \mathbf{r} \times (m\mathbf{v})$ yields $L_z = bmv_\infty$, where $v_\infty = (2E/m)^{1/2}$ is the initial (and hence the final) velocity of the particle.

²¹ Alternatively, this result may be recovered directly from Eq. (62) whose parameters, at $E > 0$, may be expressed via the same dimensionless parameter ($2Eb/\alpha$): $p = b(2Eb/\alpha)$, $e = [1 + (2Eb/\alpha)^2]^{1/2} > 1$.

The general equation (67) and the Coulomb-specific relations (68) present a formally complete solution of the scattering problem. However, in a typical experiment on elementary particle scattering the impact parameter b of a single particle is random and unknown. In this case, our results may be used to obtain *statistics* of the scattering angle θ , in particular the so-called *differential cross-section*²²

$$\frac{d\sigma}{d\Omega} \equiv \frac{1}{n} \frac{dN}{d\Omega}, \quad (3.69)$$

where n is the average number of the incident particles per unit area, and dN is the average number of particles scattered into a small solid angle range $d\Omega$. For a spherically-symmetric scattering center, which provides the axially-symmetric scattering pattern, we can take calculate $d\sigma/d\Omega$ by counting the number of incident particles within a small range db of the impact parameter:

$$dN = n 2\pi b db. \quad (3.70)$$

and hence scattered into the corresponding small range $d\Omega = 2\pi \sin\theta d\theta$. Plugging these relations into Eq. (69), we get the following general geometric relation:

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|. \quad (3.71)$$

In particular, for the Coulomb potential (49), a straightforward differentiation of Eq. (68) yields the so-called *Rutherford scattering formula*

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E} \right)^2 \frac{1}{\sin^4(\theta/2)}. \quad (3.72)$$

This result, which shows very strong scattering to small angles (so strong that the integral which expresses the full cross-section σ is formally diverging at $\theta \rightarrow 0$),²³ and weak *backscattering* (scattering to angles $\theta \approx \pi$) was historically extremely significant: in the early 1910s its good agreement with α -particle scattering experiments carried out by E. Rutherford's group gave a strong justification for "planetary" models of atoms, with electrons moving about very small nuclei.

Note that elementary particle scattering is frequently accompanied with electromagnetic radiation and/or other processes leading to the loss of the initial mechanical energy of the system, leading to *inelastic scattering*, which may give significantly different results. (In particular, a capture of an incoming particle becomes possible even for a Coulomb attracting center.) Also, quantum-mechanical effects may be important at scattering, so that the above results should be used with caution.

²² This terminology stems from the fact that an integral of $d\sigma/d\Omega$ over the full solid angle, called the *full cross-section* σ , has the dimension of area: $\sigma = N/n$, where N is the total number of scattered particles.

²³ This divergence, which persists at the quantum-mechanical treatment of the problem, is due to particles with large values of b , and disappears at an account, for example, for a finite concentration of the scattering centers.