Chapter 4. Oscillations

In this course, oscillations in 1D (and effectively 1D) systems will be discussed in quite detail, because of key importance of these phenomena for physics and engineering. We will start with the so-called "linear" oscillators whose differential equations allow full analytical solutions, and then proceed to "nonlinear" and "parametric" systems whose dynamics may be only explored by either approximate analytical or numerical methods.

4.1. Forced and damped oscillations

In Sec. 3.2 we briefly reviewed oscillations in a very important Hamiltonian system: a "harmonic oscillator" described by a 1D Lagrangian[1]

$$L \equiv T(\dot{q}) - U(q) = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2, \quad \text{with } \frac{\kappa}{m} \equiv \omega_0^2 > 0, \tag{4.1}$$

whose equation of motion in the autonomous case,

$$m\ddot{q} + \kappa x = 0, \tag{4.2}$$

has the general solution similar to Eq. (3.16):

$$q(t) = q_{\text{free}}(t) = u\cos\omega_0 t + v\sin\omega_0 t = A\cos(\omega_0 t - \varphi), \quad u = A\cos\varphi, \quad v = A\sin\varphi, \tag{4.3a}$$

where A is the amplitude and φ the phase of the oscillations which are determined by the initial conditions. Mathematically, it is frequently easier to work with sinusoidal functions in their complex-exponential form, taking[2]

$$q_{\text{free}}(t) = \text{Re}\left[Ae^{-i(\omega_0 t - \varphi)}\right] = \text{Re}\left[ae^{-i\omega_0 t}\right], \tag{4.3b}$$

where a the *complex amplitude*:

$$a \equiv Ae^{i\varphi}, \quad |a| = A, \quad \text{Re}\, a = A\cos\varphi = u, \quad \text{Im}\, a = A\sin\varphi = v. \tag{4.4}$$

Equations (3) represent the so-called "*free oscillations*" of the system. Now let us now study its more general motions. First, an additional external force $F(t)$ may induce *forced oscillations* in this system. This process is described by a linear but *inhomogeneous* equation

$$m\ddot{q} + \kappa q = F(t), \tag{4.5a}$$

or, more conveniently,

$$\ddot{q} + \omega_0^2 q = f(t), \quad f(t) \equiv F(t)/m. \tag{4.5b}$$

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[1] For the notation simplicity, in this chapter I will drop indices "ef" in the energy components T and U, and parameters like m, κ, etc. However, the reader should remember that our results will be valid for *any* 1D dynamic systems with similar equations of motion, regardless of its physics nature.

[2] Note that this is the so-called physics notation. Most engineering texts use the opposite sign in the imaginary exponent, with numerous sign implications for intermediate formulas, but (of course) similar final results.

For a particle of mass *m*, confined to a straight line, such equation is just an expression of the 2nd Newton law (or rather one of its Cartesian component). In the more general case, Eq. (5) arises in any case when the potential energy of the system has an additional time-dependent term which is linear in the generalized coordinate:

$$U(q) = \frac{\kappa q^2}{2} + \text{const} \to U(q,t) = \frac{\kappa q^2}{2} - F(t)q + \text{const}. \tag{4.6}$$

The linear differential equation (5) may be approached by two (mathematically equivalent) methods whose relative convenience depends on the character of function $f(t)$.

(i) Frequency domain. Let us present function $F(t)$ as a Fourier sum of sinusoidal harmonics:

$$f(t) = \text{Re}\sum_{\omega} f_{\omega} e^{-i\omega t} \quad . \tag{4.7}$$

Then, due to linearity of Eq. (5), its general solution may be presented as a sum of free oscillations (3) with eigenfrequency ω_0, and forced oscillations due to each of the Fourier components of force:[3]

$$q(t) = q_{\text{free}}(t) + q_{\text{forced}}(t), \quad q_{\text{forced}}(t) = \text{Re}\sum_{\omega} q_{\omega}(t), \tag{4.8}$$

where q_ω is the periodic solution to the following differential equation

$$\ddot{q}_{\omega} + \omega_0^2 q_{\omega} = f_{\omega} e^{-i\omega t}. \tag{4.9}$$

Let us look for the solution in the natural form, similar to Eq. (3):

$$q_{\omega}(t) = a_{\omega} e^{-i\omega t}. \tag{4.10}$$

Plugging Eq. (10) into Eq. (9),

$$\left(-\omega^2 + \omega_0^2\right) a_{\omega} e^{-i\omega t} = f_{\omega} e^{-i\omega t}, \tag{4.11}$$

and requiring the factors before $\exp\{-i\omega t\}$ in both parts to be equal, we readily get the well-known expression,

$$a_{\omega} = f_{\omega} \frac{1}{\omega_0^2 - \omega^2}, \tag{4.12}$$

which describes, in particular, a sharp increase of the real oscillation amplitude $A_\omega = |a_\omega|$ at $\omega \to \omega_0$ - see the black line (marked as "$Q = \infty$") in Fig. 1. This is the famous phenomenon of *resonance*, so ubiquitous in physics. Notice the fact which will be very useful for the of more complex cases (see Sec. 2 below): the solution described by Eq. (8) diverges at the exact resonance ($\omega = \omega_0$).[4]

This singularity disappears at the account of even small energy losses in the system. The simplest model of such losses is presented by the following linear equation of motion:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = f(t), \tag{4.13}$$

[3] In physics, this mathematical property of linear equations is frequently called the *linear superposition principle*.
[4] For the discussion of oscillator dynamics in this special case, see Problem 3.

with positive parameter δ called the *damping coefficient*..

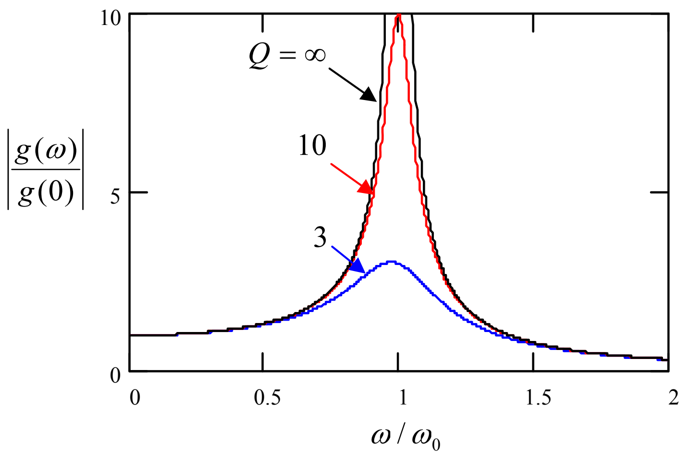

Fig. 4.1. Resonance in a harmonic oscillator for several values of the Q-factor.

Since at $\delta \neq 0$ the system is not Hamiltonian, Eq. (13) cannot be obtained from the canonical Lagrange equation (2.19). However, it follows from its general version (2.16) if the generalized force has a non-conservative component $(\mathscr{F}_q)_\delta = -2\delta m\dot{q}$. In most mechanical systems, the addition of such linear dissipative force is a phenomenological operation. However, in statistical mechanics such dissipative term may be derived, for many systems, as an *average* force exerted on a body by its *environment* whose numerous parts (described by numerous degrees of freedom different from q) are in random, though possibly thermodynamically-equilibrium states.[5]

Repeating our frequency-domain analysis of forced sinusoidal oscillations for Eq. (13), we see that Eq. (12) should be generalized as follows:

$$a_\omega = f_\omega g(\omega), \quad \text{with } g(\omega) = \frac{1}{\left(\omega_0^2 - \omega^2\right) - 2i\omega\delta}, \tag{4.14}$$

so that the real amplitude of oscillations

$$A_\omega \equiv |a_\omega| = |f_\omega||g(\omega)|, \qquad |g(\omega)| = \frac{1}{\left[(\omega_0^2 - \omega^2)^2 + (2\omega\delta)^2\right]^{1/2}}. \tag{4.15}$$

This result shows that dissipation smoothes the resonance peak – see Fig. 1 which is plotted for three values of the so-called *Q-factor*

$$Q \equiv \frac{\omega_0}{2\delta}. \tag{4.16}$$

The dimensionless factor Q allows for several interpretations. First, as an elementary analysis of Eq. (15) shows, Q gives the ratio of the oscillator response magnitudes at $\omega = \omega_0$ and $\omega = 0$ - see Fig. 1.

[5] Since such force also has a random component, dissipation is fundamentally related to *fluctuations*, and the latter effects may be neglected (as they are in this course) only if the oscillation energy is much higher than the energy scale of random fluctuations of the environment (in the thermal equilibrium at temperature T, the larger of k_BT and $\hbar\omega_0$) - see, e.g., SM Ch. 5 and QM Ch. 6.

Second, for the most interesting case of *weak damping* ($\delta << \omega_0$, i.e. $Q >> 1$), the reciprocal Q-factor gives the normalized value of the so-called *full-width at half-maximum* (FWHM) of the resonance curve:

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}. \tag{4.17}$$

Indeed, $\Delta\omega$ is defined as the difference ($\omega_2 - \omega_1$) between the two values of ω at which the square of oscillator response function, $|g(\omega)|^2$ (which is in particular, a measure of oscillation energy), is a half of its resonance value. In the weak damping limit, both these points are very close to ω_0, so that with a good accuracy we can approximate ($\omega_0^2 - \omega^2$) as either ($-2\omega\xi$) or ($-2\omega_0\xi$), where

$$\xi \equiv \omega - \omega_0 \tag{4.18}$$

is a very convenient parameter called *detuning*. (We will repeatedly use it in this course.) In this approximation, the second of Eqs. (15) is reduced to

$$|g(\omega)|^2 = \frac{1}{4\omega^2\left(\delta^2 + \xi^2\right)}. \tag{4.19}$$

As a result, points $\omega_{1,2}$ correspond to $\xi^2 = \delta^2$, i.e. $\omega_{2,1} = \omega_0 \pm \delta = \omega_0 (1 \pm 1/2Q)$, so that $\omega_2 - \omega_1 = \omega_0/Q$, thus proving Eq. (17).

Finally, one more convenient definition of Q comes from the analysis of the free oscillation *decay* which takes place at finite damping, $\delta > 0$ (which we would need to carry out anyway). Let us find the law of that decay from Eq. (13) with $f \equiv 0$, so it becomes a homogeneous, linear differential equation of the second order. As was discussed in Sec. 3.2, the general solution of such has the form of the sum (3.13) of two exponents of the type $\exp\{\lambda t\}$, with arbitrary coefficients. Plugging an arbitrary exponent, , into the homogeneous equation, we get the following algebraic characteristic equation for λ:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0. \tag{4.20}$$

Solving this quadratic equation,

$$\lambda_\pm = -\delta \pm i\omega_0', \qquad \omega_0' \equiv \left(\omega_0^2 - \delta^2\right)^{1/2}, \tag{4.21}$$

for not very high damping ($\delta < \omega_0$, i.e. $Q > 1/2$)[6] we get

$$q(t) = q_{\text{free}}(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t} = \left(u_0 \cos\omega_0' t + v_0 \sin\omega_0' t\right) e^{-\delta t} = A_0 e^{-\delta t}\cos\left(\omega_0' t - \varphi_0\right). \tag{4.22}$$

The result shows that, besides some re-normalization of the free oscillation frequency (which is negligible at weak damping), the energy dissipation leads to an exponential decay of oscillation amplitude with time constant $1/\delta$. For the oscillation energy, which scales as amplitude squared, the time constant is twice shorter:

[6] Systems with very high damping ($Q > ½$) can hardly be called oscillators, and though they are used in engineering and experimental practice (e.g., for shock, vibration and sound isolation), I have to refer the reader to special literature for their discussion – see, e.g., C. Harris and A. Piersol, *Shock and Vibration Handbook*, 5th ed., McGraw Hill, 2002.

$$\tau = \frac{1}{2\delta} = \frac{Q}{\omega_0}. \qquad (4.23)$$

This equation gives one more interpretation, and simultaneously one more effective means for experimental measurement of the Q-factor.[7]

(ii) Time domain. Returning to the forced oscillations, one may argue that the set of Eqs. (8), (10) and (14) provide a full and simple solution to this problem, which is especially simple at $t \gg \tau$ when the free oscillations have decayed. For a sinusoidal external force, this is certainly true, but real forces may be much more complicated functions of time. In this case, we should first calculate the complex amplitudes f_ω participating in the Fourier sum (7). In the general case of non-periodic $f(t)$, this is actually the Fourier integral, so that f_ω should be calculated using the reciprocal Fourier transform,[8]

$$f_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t')e^{i\omega t'}dt'. \qquad (4.24)$$

Now we can use Eq. (14) for each Fourier component of the resulting oscillations, and rewrite the final result as

$$\begin{aligned} q_{\text{forced}}(t) &= \int_{-\infty}^{+\infty} a_\omega e^{-i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, g(\omega) \int_{-\infty}^{+\infty} dt'\, f(t')e^{i\omega(t'-t)} \\ &= \int_{-\infty}^{+\infty} dt'\, f(t') \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, g(\omega)e^{i\omega(t'-t)}\right], \end{aligned} \qquad (4.25)$$

with the response function $g(\omega)$ given by Eq. (12). Besides requiring two integrations, Eq. (25) is conceptually uncomforting: it seems to indicate that the oscillator position at time t depends not only on the external force exerted at earlier times $t' < t$, but also in future. This would contradict one of the most fundamental principles of physics (and indeed, science as a whole), the *causality*: no effect may precede its cause.

Fortunately, the response function (14), and, mercifully, response functions of all real physical systems satisfy the following rule:[9]

$$\int_{-\infty}^{+\infty} g(\omega)e^{-i\omega\tau} d\omega = 0, \qquad \text{for } \tau < 0. \qquad (4.26)$$

This allows us to write the last form of Eq. (25) in either of two equivalent forms:

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7 The range of Q in important oscillators is very broad, all the way from $Q \sim 10$ for a human leg (with relaxed muscles) to $Q \sim 10^{12}$ for carefully designed optical and microwave cavities.

8 The Re operation, used in Eq. (7), may now be dropped, because for any physical (real) force, the imaginary components should compensate each other. This imposes the following condition of the complex Fourier amplitudes: $F(-\omega) = F^*(\omega)$, where the star means the complex conjugation.

9 Following tradition, the frequency-domain expression of this fact (the so-called *Kramers-Kronig relations* for response functions), will be discussed in EM Sec. 7.4.

$$q_{\text{forced}}(t) = \int_{-\infty}^{t} f(t')G(t-t')dt' = \int_{0}^{\infty} f(t-\tau)G(\tau)d\tau \,, \tag{4.27}$$

where $G(\tau)$, defined as the Fourier transform of the response function,

$$G(\tau) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\omega) e^{-i\omega\tau} d\omega \tag{4.28}$$

(and, according to Eq. (24), equal to zero for all $\tau < 0$), is called the (*temporal*) *Green's function* of the system.

While the second form of Eq. (27) is more convenient for applications, its first form is more clear conceptually. Namely, it expresses the linear superposition principle in time domain, and may be interpreted as follows: the full effect of force $f(t)$ on an oscillator (actually, any "linear system"[10]) may be described as a sum of effects of short pulses of duration dt' and amplitude $F(t')$:

$$q_{\text{forced}}(t) = \lim_{\Delta t' \to 0} \sum_{t'=-\infty}^{t} G(t-t') f(t') \Delta t'. \tag{4.29}$$

- see Fig. 2. The Green's function $G(\tau)$ thus describes the oscillator response to a unit pulse of force, measured at time $\tau = t - t'$ after the pulse.

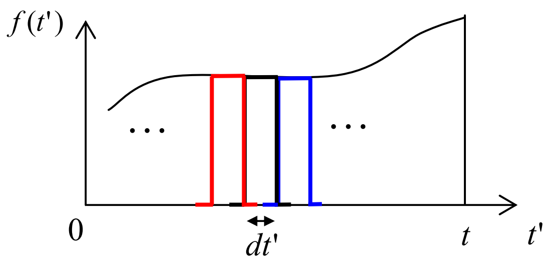

Fig. 4.2. Presentation of the force function as a sum of short pulses.

Mathematically, it is more convenient to go to the limit $dt' \to 0$ and describe the elementary, unit-area pulse by Dirac's δ-function,[11] thus returning to Eq. (27). This line of reasoning also gives a convenient way of calculation of the Green's function. Indeed, for the particular case,

$$f(t) = \delta(t - t_0), \quad \text{with } t_0 < t, \tag{4.30}$$

Eq. (27) yields $q(t) = G(t - t_0)$. Hence the Green's function may be calculated as a solution of the differential equation of motion of the system (in our case, Eq. (13)) with the δ-functional right-hand part:

$$\frac{d^2 G(\tau)}{d\tau^2} + 2\delta \frac{dG(\tau)}{d\tau} + \omega_0^2 G(\tau) = \delta(\tau), \tag{4.31}$$

and zero initial conditions:

[10] This is a very unfortunate, but common jargon, meaning "the system described by linear equations of motion".

[11] For a reminder of the basic properties of the δ-function, see MA Sec. 14.

$$G(-0) = \frac{dG}{d\tau}(-0) = 0. \tag{4.32}$$

This equation may be simplified even further. Let us integrate both sides of Eq. (31) over a infinitesimal interval including the origin, e.g., [$-d\tau/2$, $+d\tau/2$] (formally, [-0, +0]). Since the Green's function and its first derivative have to be continuous because of their physical sense, all terms in the left hand part but the first one vanish, while the first term yields $dG/d\tau|_{+0} - dG/d\tau|_{-0}$. Due to second of Eqs. (32), the last of these two terms equals zero, while right-hand part yields 1. Thus, $G(\tau)$ may be calculated for $\tau > 0$ (i.e. for all times when the function is different from zero) by solving the *homogeneous* version of the equation of motion of the system for $\tau > 0$, with the following special initial conditions:

$$G(0) = 0, \quad \frac{dG}{d\tau}(0) = 1. \tag{4.33}$$

This approach gives us a convenient way for calculation of the Green's function of a broad range of linear systems. In particular for the oscillator with not very low damping ($Q > ½$), imposing boundary conditions (33) on the general solution (22), we immediately get

$$G(\tau) = \frac{1}{\omega_0'} e^{-\delta\tau} \sin \omega_0' \tau. \tag{4.34}$$

(We could of course get the same result from Eq. (28) with the response function $g(\omega)$ given by Eq. (14), but that way is evidently more cumbersome.)

Equations (27) and (34) provide a very convenient recipe for solving most forced oscillations problems. As a simple example, let us calculate the transient process in a damped oscillator under the effect of a constant force being turned on at $t = 0$:

$$f(t) = \begin{cases} 0, & t < 0, \\ f_0, & t > 0, \end{cases} \tag{4.35}$$

provided that at $t < 0$ the oscillator was at rest (so that $q_{\text{free}}(t) \equiv 0$). Then the second form of Eq. (27) yields

$$q(t) = \int_0^\infty f(t-\tau)G(\tau)d\tau = f_0 \int_0^t \frac{1}{\omega_0'} e^{-\delta\tau} \sin \omega_0' \tau d\tau. \tag{4.36}$$

The simplest way to work out this integral is to present the sine as the imaginary part of the exponent of the same argument, giving

$$q(t) = f_0 \frac{1}{\omega_0'} \text{Im}\left[\frac{1}{\delta + i\omega_0'} e^{-\delta\tau - i\omega_0'\tau}\right]_0^t = \frac{F_0}{k}\left[1 - e^{-\delta t}\left(\cos \omega_0' t + \frac{\delta}{\omega_0'} \sin \omega_0' t\right)\right]. \tag{4.37}$$

This result, plotted in Fig. 3, is natural: it describes nothing more than the transient from the initial equilibrium position $q = 0$ to the new equilibrium position $q_0 = f_0/\omega_0^2 = F_0/k$, accompanied by decaying oscillations. For this particular simple function $f(t)$, the same result might be also obtained by introducing a new variable $\tilde{q} \equiv q - q_0$ and solving the resulting *homogeneous* equation for $\tilde{q}$ (with appropriate initial conditions), but for more complicated functions $f(t)$ the Green's function approach is indispensable.

Note that for any particular system the Green's function should be calculated only once, and then may be repeatedly used in Eq. (27) to calculate the system response to various external forces - either analytically or numerically. This property makes the Green's function approach very popular in many other fields of physics (with the corresponding generalization or re-definition of the function).

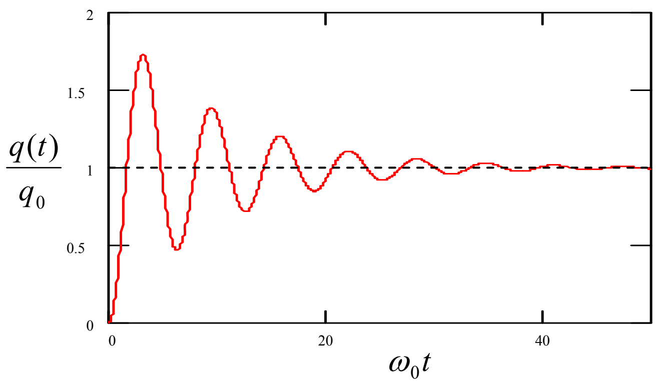

Fig. 4.3. Transient process in a linear oscillator, induced by a step-like force $f(t)$, for the particular case $\delta/\omega_0 = 0.1$.

4.2. Weakly nonlinear oscillations

In comparison with systems discussed in the last section, which are described by linear differential equations with constant coefficients and thus allows a complete and exact analytical solution, oscillations in nonlinear systems generally present a complex and analytically intractable problem. Let us start a discussion of such "*nonlinear oscillations*"[12] from an important case of which may be explored analytically.

In many 1D oscillators, higher terms in the potential expansion (3.8) cannot be neglected, but are small and may be accounted for approximately. If, in addition, damping is low (or negligible), the equation of motion may be presented as

$$\ddot{q} + \omega^2 q = f(t, q, \dot{q}, ...), \tag{4.38}$$

where the right-hand part is small (say, scales as some small dimensionless parameter $\varepsilon << 1$) and may be considered as a *perturbation*. Since at $\varepsilon = 0$ this equation has the sinusoidal solution given by Eq. (3), one might think that at nonvanishing but small ε, the approximate solution to Eq. (38) should be sought in the form

$$q(t) = q^{(0)} + q^{(1)} + q^{(2)} + ..., \quad q^{(n)} \propto \varepsilon^n, \tag{4.39}$$

with $q^{(0)} = A \cos(\omega t - \varphi) \propto \varepsilon^0$.

This is a wonderful example of an apparently impeccable mathematical reasoning which would lead to a very inefficient procedure. Indeed, let us apply it to the problem we already know the exact

[12] Again, "*nonlinear oscillations*", as well as "*parametric oscillations*" discussed below, are unfortunate but generally accepted jargon terms. It would be certainly more reasonable to speak about oscillations in nonlinear (or parametric) *systems*.

solution for, namely the free oscillations in a linear damped oscillator, for this occasion considering the damping very small. The corresponding equation of motion, Eq. (13) with $f(t) = 0$, may be presented in form (38) if we take $\omega = \omega_0$ and

$$f = -2\delta\dot{q}, \quad \delta \propto \varepsilon . \tag{4.40}$$

The naïve approach described above would allow us to find corrections, of the order of δ, to the free, non-decaying oscillations $A\cos(\omega_0 t - \varphi)$. However, we already know that the main effect of damping is the gradual decrease of the oscillation amplitude to zero – see Eq. (22), though at low damping, $\delta << \omega$, this decay happens at a large time scale $t \sim \tau >> 1/\omega_0$. Hence, if we want our approximate method to be productive (i.e. to work at all time scales), we need to account for the fact the small RHP of Eq. (38) may lead to essential changes of amplitude A (and, as we will see below, also of phase φ) at large times, because of the *slowly accumulating* effects of the small RHP perturbation.

This goal may be achieved by the account of these changes of the "basic" part of the solution in expansion (39):

$$q^{(0)} = A(t)\cos[\omega t - \varphi(t)], \quad \text{with } \dot{A}, \dot{\varphi} \to 0 \quad \text{at } \varepsilon \to 0. \tag{4.41}$$

Approximate methods, based on Eqs. (39) and (37), have several varieties and several names,[13] but their basic idea (and the results in the first approximation in ε) are the same. Let us study this approach on a simple but very representative example of a dissipative (but high-Q) pendulum driven by weak sinusoidal external force with a nearly-resonant frequency:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 \sin q = f_0 \cos\omega t, \tag{4.42}$$

with $|\omega - \omega_0|$, $\delta << \omega_0$, and f_0 so small that $|q| << 1$ at all times. Expanding $\sin q$ into the Taylor series, keeping only the first two terms of this expansion, and moving all the small terms to the right-hand part, we can bring Eq.(38) to the form:

$$\ddot{q} + \omega^2 q = -2\delta\dot{q} + 2\xi\omega q + \alpha x^3 + f_0\cos\omega t \equiv f(t, x, \dot{x}). \tag{4.43}$$

Here $\alpha = \omega_0^2/6$ in the case of the pendulum, but the calculations below will be valid for any α. Equation (43) differs from the "canonical" form Eq. (34) only by one feature: anticipating the actual solution (describing forced oscillations) to have frequency ω rather than ω_0, we have moved the difference between $\omega^2 q$ and $\omega_0^2 q$ into the right-hand part, using the notion of detuning $\xi \equiv \omega - \omega_0 \approx (\omega^2 - \omega_0^2)/2\omega$ which we already used earlier – see Eq. (13).

Now, following the general recipe expressed by Eqs. (39) and (41), in the first approximation in $f_m \propto \varepsilon$, we may look for the solution to Eq. (43) in the form

$$q(t) = A\cos\Psi + q^{(1)}(t), \quad \Psi \equiv \omega t - \varphi, \qquad x^{(1)} \sim \varepsilon . \tag{4.44}$$

Let us plug in this assumed solution into both parts of Eq. (43), leaving only the terms of the first order in ε. Thanks to our trick with detuning, the two zero-order terms in the left-hand part cancel each other. Moreover, since each RHP term of Eq. (43) is already of the order of ε, we can drop $q^{(1)} \propto \varepsilon$ from the

[13] Depending of the text, one can meet references to either *small parameter methods* (the name I will use), or *asymptotic methods*, or (especially in nonlinear and laser optics) the *rotating wave approximation* (RWA). The names of founding fathers of the field, including J. H. Poincaré, B. van der Pol, N. Krylov, N. Bogolyubov, and Yu. Mitroplolsky are also frequently associated with this method and its variations.

substitution into that part at all, because this would give us only terms $O(\varepsilon^2)$ or higher. As a result, we get the following approximate equation:

$$\ddot{q}^{(1)} + \omega^2 q^{(1)} = f^{(0)} \equiv -2\delta \frac{d}{dt}(A\cos\Psi) + 2\xi\omega A\cos\Psi + \alpha\left(A\cos\Psi\right)^3 + f_0\cos\omega t. \qquad (4.45)$$

According to Eq. (41), generally A and φ should be considered as (slow) functions of time. However, let us leave the analyses of transient process and system stability until the next section, and use Eq. (45) to find stationary oscillations in the system, which are established after the initial transient. For that limited task, we may accept A = const, φ = const, so that $q^{(0)}$ presents sinusoidal oscillations of frequency ω. Sorting the terms in the RHP according to their time dependence,[14] we see that it has terms with that frequency and also a term with frequency 3ω:

$$f^{(0)} = (2\xi\omega A + \frac{3}{4}\alpha A^3 + f_0\cos\varphi)\cos\Psi + (2\delta\omega A - f_0\sin\varphi)\sin\Psi + \frac{1}{4}\alpha A^3\cos 3\Psi. \qquad (4.46)$$

Now comes the main trick of the small parameter method: mathematically, Eq. (45) is the equation of oscillations in a loss-free harmonic oscillator of frequency ω (not ω_0!) under the action of an external force having three terms: two "quadrature" components at that very frequency ω, and the third one at frequency 3ω. As we know from our analysis of this problem in Sec. 1, if any of the first two components is nonvanishing, $q^{(1)}$ grows to infinity – see Eq. (12). At the same time, by the very structure of the small parameter method, $q^{(1)}$ has to be finite (moreover, small – see Eq. (39)). The only way out of this contradiction is to require that amplitudes of both quadrature components of frequency ω are equal to zero:

$$2\xi\omega A + \frac{3}{4}\alpha A^3 + f_0\cos\varphi = 0, \quad 2\delta\omega A - f_0\sin\varphi = 0. \qquad (4.47)$$

These two "*harmonic balance*" equations enable us to find two parameters of the forced oscillations: amplitude A and phase φ. In particular, the phase may be readily excluded from this system, and the solution for amplitude A presented in the following implicit but convenient form:

$$A^2 = \frac{f_0^2}{4\omega^2}\frac{1}{\xi^2(A) + \delta^2}, \qquad \xi(A) \equiv \xi + \frac{3}{8}\frac{\alpha A^2}{\omega} = \omega - \left(\omega_0 - \frac{3}{8}\frac{\alpha A^2}{\omega}\right). \qquad (4.48)$$

This expression differs from Eq. (19) for the linear resonance only by using the small-damping approximation (10) and (more substantially) by the replacement of the eigenfrequency ω_0 of the resonator for its effective, amplitude-dependent value

$$\omega_0(A) = \omega_0 - \frac{3}{8\omega}\alpha A^2. \qquad (4.49)$$

The physical meaning of $\omega_0(A)$ is simple: this is just the frequency of free oscillations of amplitude A in the same system at zero damping. Indeed, for $\delta = 0$ and $f_0 = 0$ we could repeat our calculations, assuming that ω is an amplitude-dependent eigenfrequency $\omega_0(A)$, to be found. Then the second of Eqs. (47) is trivially satisfied, while the second of them gives Eq. (49).

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14 Using the second of Eqs. (44), I have presented cos ωt as cos $(\Psi + \varphi) \equiv$ cos Ψ cos φ - sinΨ sin φ, and used the well-known trigonometric identity $\cos^3\Psi = (3/4)\cos\Psi + (1/4)\cos 3\Psi$ - see, e.g., MA Eq. (3.3).

Expression (48) allows one to draw the curves of this "*nonlinear resonance*" just by curving the linear resonance plots (Fig. 1) according to the "*skeleton curve*" expressed by Eq. (49). Figure 4 shows the result of this procedure. Note that at small amplitude we return to the usual, "*linear*" resonance.

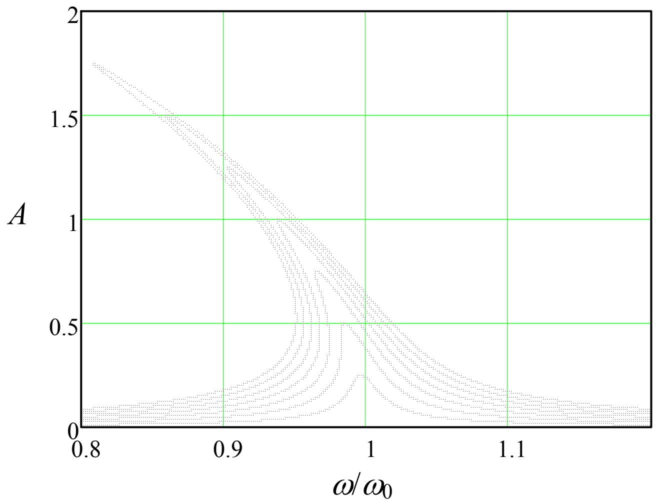

Fig. 4.4.Nonlinear resonance as described by the small parameter method - see Eq. (48) - for the case $\alpha = \omega_0^2/6$, $\delta/\omega = 0.01$ (i.e. $Q = 50$), and 7 values of parameter f_0/ω_0^2, increased from 0 to 0.035 by equal steps.

To bring our solution to the logical completion, we should still find the first perturbation $q^{(1)}(t)$ from what is left of Eq. (45). Since the structure of this equation is similar to Eq. (9) with the force of frequency 3ω, we may re-write Eq. (12) to obtain

$$q^{(1)}(t) = -\frac{1}{32\omega^2}\alpha A^3 \cos 3(\omega t - \varphi). \qquad (4.50)$$

(In the spirit of the small parameter method, we have neglected small terms of the order of $(\delta/\omega)^2$ and $(\delta/\omega)^2$, both of the order of ε^2.) This result (notice the negative sign!) shows that as the amplitude of an oscillator with $\alpha > 0$ (e.g., a pendulum) grows, the waveforms become a bit more "blunt" near the maximum deviations from the equilibrium. Equation (50) also allows us to estimate the validity range of the small parameter method. Since it has been based on the assumption $|q^{(1)}| << |q^{(0)}| \leq A$, for this particular problem we have to require $\alpha A^2/32\omega^2 << 1$.

From the mathematical viewpoint, the next step would be to write down the next approximation

$$q(t) = A\cos\Psi + x^{(1)}(t) + x^{(2)}(t), \qquad x^{(2)} \sim \varepsilon^2, \qquad (4.51)$$

and plug it into Eq. (43) which (thanks to our special choice of $q^{(0)}$ and $q^{(1)}$), would retain only $\ddot{q}^{(2)} + \omega^2 q^{(2)}$ in its left-hand part. Again, requiring that amplitudes of two quadrature components of frequency ω in the RHP to be zero, we may get the second-order corrections to A and φ.. Then we may use the remaining part of the equation to calculate $q^{(2)}$, and then go after the third-order terms, etc. [15] However, for most purposes the sum $q^{(0)} + q^{(1)}$, and sometimes even just the crudest approximation $q^{(0)}$ alone is completely sufficient. For example, according to Eq. (50), for a simple pendulum ($\alpha = \omega_0^2/6$) swinging as much as between the opposite horizontal positions ($A = \pi/2$), the 1st order correction $q^{(1)}$ is of the order of 0.5%. (Soon beyond this value, completely new dynamic phenomena start – see Sec. 7

[15] For a mathematically rigorous treatment of the higher approximations, see, e.g., Yu. A. Mitropolsky and N. V. Dao, *Applied Asymptotic Methods in Nonlinear Oscillations*, Springer, 2004.

below. These phenomena cannot be covered by the small parameter theory at all.) Due to this reason, higher approximations are rarely used in practical physics and engineering.

A much more important issue is the stability of solutions described by Eq. (48). Indeed, Fig. 4 shows that in some range of parameters, these equations give three different values for the oscillation amplitude (and phase), and it is important to understand which of these solutions are stable. Since these solutions are not the fixed points in the sense discussed in the Sec. 3.2 (each of the points present a nearly-sinusoidal oscillation), a different approach is needed for their stability analysis. Such approach is provided by the so-called *van der Pol method* which is a generalization of the small parameter method (in the first approximation in ε) to the case of the amplitude and phase evolving in time. This method will also enable us to analyze non-stationary (especially initial, "*transient*") processes which are of key importance for some dynamic systems.

4.3. The van der Pol method

First of all, let us formulate the way the harmonic balance equations, such as Eq. (47), should be obtained for the general case (38). After plugging in the 0^{th} approximation (37) into the right-hand part of equation (38) we have to require amplitudes of its both quadrature components of frequency ω to be zero. From the standard Fourier analysis we know that these requirements may be presented as

$$\frac{1}{2\pi}\int_0^{2\pi} f^{(0)} \begin{Bmatrix} \cos\Psi \\ \sin\Psi \end{Bmatrix} d\Psi = 0\,, \qquad (4.52)$$

or equivalently,

$$\overline{f^{(0)}\sin\Psi} = 0, \quad \overline{f^{(0)}\cos\Psi} = 0, \qquad (4.53)$$

where symbol $\overline{...}$ means averaging over the period $2\pi/\omega$ of the time dependence of the right-hand part of Eq. (52), with the arguments calculated in the 0-th approximation:

$$f^{(0)} \equiv f(t, x^{(0)}, \dot{x}^{(0)},...) = f\big(t, A\cos(\omega t-\varphi), -A\omega\sin(\omega t-\varphi),...\big). \qquad (4.54)$$

Now, for a transient process the contribution of $q^{(0)}$ to left-hand part of Eq. (38) is not zero any longer, because both amplitude and phase may be slow functions of time – see Eq. (41). Let us calculate this contribution in the first approximation in ε, neglecting the second derivatives of A and φ:

$$\ddot{q}^{(0)} + \omega^2 q^{(0)} = \left(\frac{d^2}{dt^2}+\omega^2\right)A(t)\cos\big(\omega t-\varphi(t)\big) \approx -2\dot{A}\omega\sin(\omega t-\varphi) + 2A\dot{\varphi}\omega\cos(\omega t-\varphi). \qquad (4.55)$$

In contrast, in the right-hand part of Eq. (52), we can neglect the time derivatives of the amplitude and phase, because this part is already proportional to the small parameter. Hence, in the first order in ε, Eq. (38) becomes

$$\ddot{q}^{(1)} + \omega^2 q^{(1)} = f^{(0)} - \big(-2\dot{A}\omega\sin\Psi + 2A\dot{\varphi}\omega\cos\Psi\big). \qquad (4.56)$$

Now, repeating the arguments of the previous section, we have to require that the right-hand part of this equation does not contain any components of frequency ω. From here, using the same notation as in Eq. (53), we get a pair of so-called *reduced equations* (also called the "*van der Pol equations*" or the "*rotating wave approximation*") for the time evolution of the amplitude and phase:

$$\dot{A} = -\frac{1}{\omega}\overline{f^{(0)}\sin\Psi}, \quad \dot{\varphi} = \frac{1}{\omega A}\overline{f^{(0)}\cos\Psi}\,. \tag{4.57a}$$

Extending definition (4) of the complex amplitude of oscillations to their slow evolution in time, $a(t) \equiv A(t)\exp\{i\varphi(t)\}$, and differentiating this relation, two equations (57a) may be also re-written in the form of one equation for a:

$$\dot{a} = \frac{i}{\omega}\overline{f^{(0)}e^{i(\Psi+\varphi)}} \equiv \frac{i}{\omega}\overline{f^{(0)}e^{i\omega t}}\,, \tag{4.57b}$$

or two equations for the real and imaginary part of $a(t) = u(t) + iv(t)$:

$$\dot{u} = -\frac{1}{\omega}\overline{f^{(0)}\sin\omega t}, \quad \dot{v} = \frac{1}{\omega}\overline{f^{(0)}\cos\omega t}\,. \tag{4.57c}$$

The first-order harmonic balance equations (53) are evidently just the particular case of the reduced equations for stationary oscillations ($\dot{A} = \dot{\varphi} = 0$), i.e. the first approximation of the small parameter method is a particular case of the van der Pol method .[16]

Superficially, the system (57a) of two coupled, first-order differential equations may look as complex than the initial, second-order differential equation (52), but actually it is much simpler. Indeed, let us spell out their right-hand part for the easy case of a linear oscillator with damping. For that, we may use, for example, Eq. (46) with $\alpha = f_0 = 0$, turning Eqs. (4.57a) into

$$\dot{A} = -\delta A, \tag{4.58a}$$

$$A\dot{\varphi} = \xi A.. \tag{4.58b}$$

The solution to Eq. (58a) gives us the same "envelope" law $A(t) = A(0)e^{-\delta t}$ which follows from the exact solution (22) of the initial differential equation, while the elementary integration of Eq. (58b) yields $\varphi(t) = \xi t + \varphi(0) = \omega t - \omega_0 t + \varphi(0)$. This means that our approximate solution,

$$q^{(0)}(t) = A(t)\cos[\omega t - \varphi(t)] = A(0)e^{-\delta t}\cos(\omega_0 t - \varphi(0)), \tag{4.59}$$

agrees with the exact Eq. (22), besides the second-order correction to the oscillation frequency. It is remarkable how nicely the reduced equations recover the proper frequency of free oscillations in this autonomous system (in which the very notion of ω is ambiguous).

The situation is different at forced oscillations. For example, for the (generally, nonlinear) oscillator described by Eq. (43) with $f_0 \neq 0$, , Eqs. (57a) yield

$$\dot{A} = -\delta A + \frac{f_0}{2\omega}\sin\varphi, \quad A\dot{\varphi} = \xi(A)A + \frac{f_0}{2\omega}\cos\varphi\,. \tag{4.60}$$

[16] One may ask why cannot we stick to the just one, most compact, complex –amplitude form (57b) of the reduced equations. The main reason is that when function $f(x,\dot{x},t)$ is nonlinear, we cannot replace its real arguments, such as $x = A\cos(\omega t - \varphi)$, with their complex-function representations (as could be done in the linear problems considered in Sec. 4.1), and need to use real variables like $\{A, \varphi\}$ or $\{u, v\}$ anyway.

Here (after a transient), amplitude and phase tend to one of stationary states given by Eqs. (48). This means that φ becomes constant, so that $q^{(0)} \to A\cos(\omega t + \text{const})$, i.e. the reduced equations automatically recover the correct frequency of the solution, equal to that of the external force.

Note that each stationary oscillation regime, with certain amplitude and phase, corresponds to a fixed point of the reduced equations, so that the stability of those fixed points determine that of the oscillations. Let us carry out such analysis for several simple systems of key importance for physics and engineering.

4.4. Self-oscillations and phase locking

The van der Pol method was developed in the late 1920s for the analysis of one more type of oscillatory motion: *self-oscillations*. Several systems, e.g., electronic rf amplifiers with positive feedback, or optical media with electron population inversion, provide convenient means for the compensation, and even overcompensation of the intrinsic energy losses in oscillators. Phenomenologically, this effect may be described as the change of sign of the damping coefficient δ from positive to negative. Since for small oscillations the equation of motion is still linear, we can use Eq. (22) to describe its general solution. This equation shows that at $\delta < 0$, even infinitesimal deviations from equilibrium (say, due to unavoidable fluctuations) lead to oscillations with exponentially growing amplitude. Of course, in any real system such growth cannot persist infinitely, and shall be limited by this or that mechanism - e.g., in our examples, by the amplifier saturation or electron population exhaustion. In many cases, the saturation effect may be described by *nonlinear damping*:

$$2\delta\dot{q} \to 2\delta\dot{q} + \beta\dot{q}^3, \tag{4.61}$$

with $\beta > 0$. Let us analyze this phenomenon, applying the van der Pol method to the corresponding homogeneous differential equation

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q + \beta\dot{q}^3 = 0. \tag{4.62}$$

Carrying out the dissipative and detuning terms to the right hand part as f, we can readily calculate the reduced equations (57a):[17]

$$\dot{A} = -\delta(A)\, A, \quad \delta(A) \equiv \delta + \frac{3}{8}\beta\omega^2 A^2. \tag{4.63a}$$

$$A\dot{\varphi} = \xi\, A.. \tag{4.63b}$$

The latter of these equations has exactly the same form as Eq. (58b) for the case of decaying oscillations (cf. Eq. (58b)) and hence shows that the self-oscillations (if they happen, i.e. if $A \neq 0$) have frequency ω_0 of the oscillator itself.

Equation (63a) is more interesting. If the initial damping δ is positive, it has only the trivial fixed point, $A_1 = 0$ (which describes the oscillator at rest), but if δ is negative, there is also another fixed point,

$$A_2 = \left[\frac{8|\delta|}{3\beta\omega^2}\right]^{1/2}, \tag{4.64}$$

[17] For that, one needs the trigonometric identity $\sin^3\Psi = (3/4)\sin\Psi - (1/4)\sin 3\Psi$ - see, e.g., MA Eq. (3.3).

which describes steady self-oscillations with a non-zero amplitude.

Let us apply the general approach discussed in Sec. 3.2, the linearization of equations of motion, to this reduced equation. For the trivial fixed point $A_1 = 0$, the linearization of Eq. (63a) is reduced to discarding the nonlinear term in the definition of the amplitude-dependent damping $\delta(A)$. The resulting linear equation evidently shows that the system's equilibrium point, $A = A_1 = 0$, is stable at $\delta > 0$ and unstable at $\delta < 0$. (We have already discussed this *self-excitation condition* above.) The linearization of Eq. (63a) near the non-trivial fixed point requires a bit more math: in the first order in $\tilde{A} \equiv A - A_2 \to 0$, we get

$$\dot{\tilde{A}} = \dot{A} = -\delta(A_2 + \tilde{A}) - \frac{3}{8}\beta\omega^2(A_2 + \tilde{A})^3 \approx -\delta\,\tilde{A} - \frac{3}{8}\beta\omega^2 3A_2^2\tilde{A} = (-\delta + 3\delta)\tilde{A} = 2\delta\tilde{A}. \qquad (4.65)$$

We see that fixed point A_2 (and hence the self-oscillation process) is stable as soon as it exists ($\delta < 0$), very much similar to the situation in our "testbed problem" (Fig. 1.6).

Now let us consider another problem: the effect of a external sinusoidal force on a self-excited oscillator. If the force is sufficiently small, its effect on the self-excitation conditions and the oscillation amplitude is negligible. However, if frequency ω of such weak force is close to the eigenfrequency ω_0 of the oscillator, it may lead to a very important effect of *phase-locking* (also called "*synchronization*"). At this effect, the oscillator frequency deviates from ω_0, and becomes equal to ω, within a certain range

$$-\Delta \le \omega - \omega_0 < +\Delta. \qquad (4.66)$$

In order to calculate the phase locking range width 2Δ, and also find the laws governing this process, we may repeat the van der Pol calculation, adding term $f_0\cos\,\omega t$ to the right-hand part of Eq. (62) – cf. Eq. (43). This addition modifies Eqs. (63) as follows:[18]

$$\dot{A} = -\delta(A)\,A + \frac{f_0}{2\omega}\sin\varphi, \qquad (4.67a)$$

$$A\dot{\varphi} = \xi\,A + \frac{f_0}{2\omega}\cos\varphi. \qquad (4.67b)$$

If the system is self-excited, and the external force is weak, its effect on the oscillation amplitude is small, and in the first approximation in f_0 we can take $A = A_2 = \text{const}$. Plugging this approximation into the latter of Eqs. (62), we get a very simple equation (which is ubiquitous in systems with phase locking, including digital electronic circuits used for that purpose):

$$\dot{\varphi} = \xi + \Delta\cos\varphi, \qquad (4.68)$$

where in our current case

$$\Delta = \frac{f_0}{2\omega A_2}. \qquad (4.69)$$

It is evident that within range $-|\Delta| < \xi < +|\Delta|$, Eq. (68) has two fixed points on each 2π-segment:[19]

[18] This result should be evident, even without calculations, from the comparison of Eqs. (60) and (63).

[19] In most physical systems, these segments (corresponding to different integers n in Eq. (64)) cannot be distinguished, but in some, they can.

$$\varphi_{\pm} = \pm \arccos\left(-\frac{\xi}{\Delta}\right) + 2\pi n . \qquad (4.70)$$

It is easy to linearize Eq. (68) near each point to analyze their stability in our usual way; however, let me use this opportunity to demonstrate another convenient way to do this in 1D systems. Let us plot the right-hand part of Eq. (63) as a function of φ - see Fig. 5. Since the positive values of this function correspond to the growth of φ, and vice versa, we may draw the arrows showing the direction of phase evolution in time. From these arrows, it is clear that one of these fixed points (for $f_0 > 0$, φ_+) is stable, while its counterpart is unstable. Hence the magnitude of Δ given by Eq. (69) is indeed the phase locking range (or rather half-range) which we wanted to find. Note that it is proportional to the amplitude of the phase locking signal - perhaps the most important feature of phase locking.

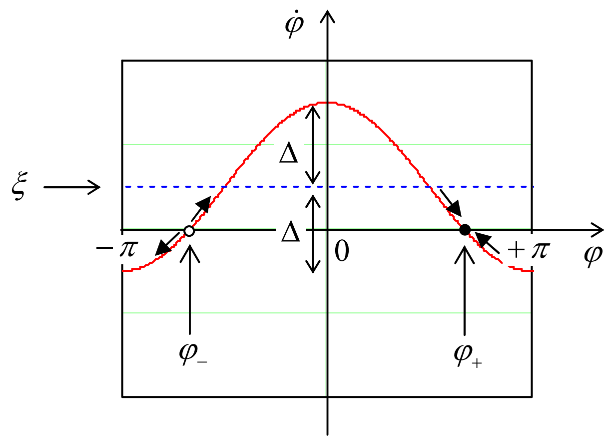

Fig. 4.5. The "phase diagram" of a phase-locked system, for the particular case $\xi = \Delta/2$, $F_0 > 0$.

In order to complete our simple analysis, based on the assumption of fixed oscillation amplitude, we need to find the condition of validity of this assumption. For that, we can linearize Eq. (67a), for the stationary case, near value (64), just as we have done in Eq. (65) for the transient process. The result,

$$\tilde{A} \equiv A - A_2 = \frac{1}{2|\delta|}\frac{f_0}{2\omega}\sin\varphi_{\pm} \approx A_2\left|\frac{\Delta}{2\delta}\right|\sin\varphi_{\pm} , \qquad (4.71)$$

shows that our assumption, $|\tilde{A}| << A_2$, and hence the final result (69), are valid if the phase locking range, 2|Δ|, is much smaller than $4|\delta|$.

4.5. Parametric excitation

In both problems solved in the last section, the stability analysis was easy because it could be carried out for just one van der Pol variable, *either* amplitude *or* phase. Generally, such analysis of the reduced equations involves both these variables. The classical example of such situation is given by one important phenomenon – the *parametric excitation* of oscillations. An elementary example of such oscillations is given by a pendulum with an externally-changed parameter, for example length $l(t)$ - see Fig. 6. Both experiments (including those with playground swings:-) and numerical simulations show that if the length is changed periodically, with frequency 2ω which is close to $2\omega_0$ and with a sufficient swing Δl, the equilibrium position of the pendulum becomes unstable, and it starts swinging with frequency ω equal *exactly* to the half of the length modulation frequency (and hence only *approximately* equal to the average eigenfrequency ω_0 of the oscillator).

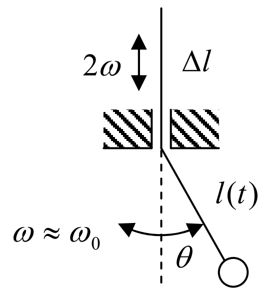

Fig. 4.6. Parametric excitation of a pendulum.

For an elementary analysis of this effect we may consider the simplest case when the changes Δl are small ($<< l$) and instant, and the oscillations are small. Then, in the lowest point ($\theta = 0$), where the pendulum moves with the highest velocity v_{max}, the string tension F is *higher* than mg by the centrifugal force: $F_{max} = mg + mv_{max}^2/l$. On the contrary, at the maximum pendulum deviation from equilibrium, the force is *weakened* by the string tilt: $F_{min} = mg\cos\theta_{max} \approx mg - mg\theta_{max}^2/2 = mg - mv_{max}^2/2l$. Thus if during each oscillation period the string is pulled up sharply at each of its two passages through the lowest point, and is let go down in each of two points of the maximum deviation, the total work per period is positive:

$$W \approx 2(F_{max} - F_{min})\Delta l = 3\frac{mv_{max}^2}{l}\Delta l = 6\frac{\Delta l}{l}E, \qquad (4.72)$$

where $E = T_{max} = mv^2_{max}/2$ is the oscillation energy. Thus, if the so-called *modulation depth* $\Delta l/l$ is sufficient, the oscillation energy may increase even if a part of it is drained out by damping $\delta > 0$. Quantitatively, Eq. (22) shows that low damping ($\delta << \omega_0$) leads to the energy decrease

$$\Delta E \approx -4\pi\frac{\delta}{\omega_0}E \qquad (4.73)$$

per oscillation period. Comparing Eqs. (72) and (73), we may write the condition of parametric excitation, for this simple case, as[20]

$$\frac{\Delta l}{l} > \frac{2\pi\delta}{3\omega_0} = \frac{\pi}{3Q}. \qquad (4.74)$$

However, this simple result does not account for the possible difference between the oscillation frequency $\omega = p/2$ and ω_0, and also does not explain whether the proper phase relation between the parametric oscillations and external force may be sustained automatically. In order to address these issues we may apply the van der Pol method to a simple but natural linear equation

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2(1 + \mu\cos 2\omega t)q = 0, \qquad (4.75)$$

[20] A modulation of pendulum's mass (say, by periodic pumping fluid into and out of a suspended balloon) gives a qualitatively similar result. Note, however, that parametric oscillations cannot be excited by modulating oscillator's *damping* (at least if it stays positive at all times), because such modulation does not change the system energy, just the energy drain rate.

describing this phenomenon for a particular case of a sinusoidal law of eigenfrequency modulation. Rewriting this equation in the canonical form

$$\ddot{q} + \omega^2 q = f(t, q, \dot{q}) = -2\delta\dot{q} + 2\xi\omega q - \mu\omega_0^2 q \cos 2\omega t, \tag{4.76}$$

and assuming that not only δ/ω and $|\xi|/\omega$, but also μ are all much less than 1, we may use general Eqs. (57a) to readily get the following reduced equations:

$$\dot{A} = -\delta A - \frac{\mu\omega}{4} A \sin 2\varphi, \tag{4.77a}$$

$$A\dot{\varphi} = \xi A - \frac{\mu\omega}{4} A \cos 2\varphi. \tag{4.77b}$$

These equations evidently have a fixed point $A_1 = 0$, but its analysis is not straightforward, because phase φ of oscillations is undetermined at such point. In order to avoid this difficulty, we may use, instead of the real amplitude and phase of oscillations, their complex amplitude $a = A \exp\{i\varphi\}$, or its Cartesian components u and v. Indeed, for our function f, Eq. (57b) gives

$$\dot{a} = (-\delta + i\xi)a - i\frac{\mu\omega}{4} a^*, \tag{4.78}$$

while Eqs. (57c) become

$$\dot{u} = -\delta u - \xi v - \frac{\mu\omega}{4} v, \tag{4.79a}$$

$$\dot{v} = -\delta v + \xi u - \frac{\mu\omega}{4} u. \tag{4.79b}$$

We see that in contrast to Eqs. (77), in Cartesian coordinates the fixed point $u_1 = v_1 = 0$ is absolutely regular. Moreover, equations (79) are already linear, so they do not require any additional linearization. Thus we can use the same approach as was already used in Secs. 3.2 and 4.1, i.e. look for the solution to Eqs. (79) in the exponential form $\exp\{\lambda t\}$. However, now we are dealing with two variables, and should allow them to have, for each value of λ, a certain ratio u/v. For that, we take the partial solution in the form

$$u = c_u e^{\lambda t}, \quad v = c_v e^{\lambda t}. \tag{4.80}$$

Plugging this solution into Eqs. (79), we get a system of two linear algebraic equations for the so-called "*distribution coefficients*" c_u and c_v:

$$\begin{aligned} (-\delta - \lambda)c_u + (-\xi - \frac{\mu\omega}{4})c_v &= 0, \\ (\xi - \frac{\mu\omega}{4})c_u + (-\delta - \lambda)c_v &= 0. \end{aligned} \tag{4.81}$$

The characteristic equation of this system,

$$\begin{vmatrix} -\delta-\lambda & -\xi-\dfrac{\mu\omega}{4} \\ \xi-\dfrac{\mu\omega}{4} & -\delta-\lambda \end{vmatrix} \equiv \lambda^2+2\delta\lambda+\delta^2+\xi^2-\left(\frac{\mu\omega}{4}\right)^2=0, \quad (4.82)$$

has two roots:

$$\lambda_\pm = -\delta \pm \left[\left(\frac{\mu\omega}{4}\right)^2-\xi^2\right]^{1/2}. \quad (4.83)$$

Requiring the fixed point to be unstable, $\mathrm{Re}\lambda_+ > 0$, we get the parametric excitation condition

$$\frac{\mu\omega}{4} > \left[\delta^2+\xi^2\right]^{1/2}. \quad (4.84)$$

Thus the parametric excitation may indeed happen without any artificial phase adjustment: the arising oscillations self-adjust their phase to pick up energy from the external source responsible for the parameter variation – see Fig. 6b.

Our key result (84) may be compared with two other calculations. First. in the case of negligible damping ($\delta = 0$), Eq. (84) turns into condition $\mu\omega/4 > |\xi|$. This result may be compared with the well-developed theory of the so-called *Mathieu equation* whose canonic form is

$$\frac{d^2y}{dv^2}+\left[a-2b\cos(2v)\right]y=0. \quad (4.85)$$

It is evident that with the substitutions $y \to q$, $v \to \omega t$, $a \to (\omega_0/\omega)^2$, $b \to -\mu/2$, this equation is just a particular case of Eq. (75) for $\delta = 0$. In terms of Eq. (84), the result of our approximate analysis may be re-written just as $q > | a - 1 |$, and is supposed to be valid for $q \ll 1$. This condition is shown in Fig. 7 together with the numerically calculated[21] stability boundaries of the Mathieu equation.

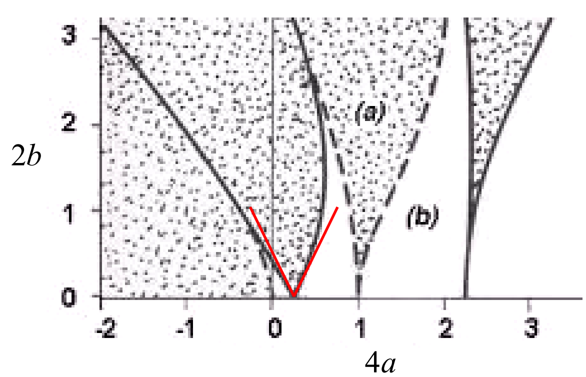

Fig. 4.7. Stability boundaries of the Mathieu equation, as calculated numerically (black lines) and by the van der Pol method (red straight lines). Dotted areas indicate the parameter regions in which the general solution to the Mathieu equation includes an exponentially growing term.

One can see that the van der Pol method results work fine within their applicability limit, though they fail to predict several other important features of the Mathieu equation, such as the existence of

[21] Such calculations may be substantially simplified by the use of the so-called *Floquet theorem*, which is also the mathematical basis for the discussion of wave propagation in periodic media – see the next chapter.

higher regions of parametric excitation (at $a \approx n^2$, i.e. $\omega_0 \approx \omega/n$, for all integer n), and a small region of stability at $a < 0$ (meaning, for example, the counter-intuitive stability of an inverted pendulum with the periodically modulated length, within a limited range of the modulation depth 2μ). The reason of these failures is the fact that, as can be seen in Fig. 7, these phenomena do not appear in the first approximation in the parameter modulation amplitude $\mu \propto q$, which is the applicability realm for the van der Pol method.

In the opposite case of finite damping but exact tuning ($\xi = 0$, $\omega \approx \omega_0$), Eq. (84) gives

$$\mu > \frac{4\delta}{\omega_0} = \frac{2}{Q}. \tag{4.86}$$

This condition may be compared with Eq. (74), taking $\Delta l/l = 2\mu$. The comparison shows that though the structure of these conditions is similar, the numerical coefficients are different. The first reason of this difference is that the instant parameter change at optimal moments of time is more efficient then the sinusoidal variation described by (75). Second, the change of pendulum length modulates not only its frequency (as Eq. (75) implies), but also its mechanical impedance $Z \equiv (gl)^{1/2}$ – see Problem 10 below.

Before moving on, let us Note three most important differences between the parametric and forced oscillations:

(i) Parametric oscillations completely disappear outside of their excitation range, while the forced oscillations have a non-zero amplitude for any frequency and amplitude of the external force.

(ii) Parametric excitation may be described by a linear homogeneous equation - e.g., Eq. (75) - which cannot predict any finite oscillation amplitude within the excitation range, even at finite damping. In order to describe stationary parametric oscillations, some nonlinear effect has to be taken into account - see, e.g., Problem 6.

(iii) Equations (77) are evidently invariant relative to the change $\varphi \to \varphi + \pi$. This is not an artifact of the van der Pol method, because the initial Eq. (75) is also invariant relative to the corresponding replacement $q(t) \to q(t - \pi/\omega)$. This invariance means in particular that the parametric oscillations may be excited with two phases which differ by π, with other properties absolutely similar. At the dawn of the computer age (in the 1950s and early 1960s), there were attempts to use this property for storage and processing digital information in the phase-binary code.

4.6. Fixed point classification

Reduced equations (79) give us a nice pretext for a brief discussion of fixed points of a general dynamic system described by two time-independent, first-order differential equations.[22] After their linearization near a fixed point, the equations for deviations can always be presented in the form similar to Eq. (79):

[22] Autonomous systems described by a single second-order differential equation, say $F(q,\dot{q},\ddot{q}) = 0$, also belong to this class, because we may treat velocity $\dot{q} \equiv v$ as a new variable, and use this definition as one first-order differential equation, and the initial equation, in the form $F(q,v,\dot{v}) = 0$, as the second first-order equation.

$$\dot{q}_1 = M_{11}q_1 + M_{12}q_2, \qquad \dot{q}_2 = M_{21}q_1 + M_{22}q_2, \tag{4.87}$$

where $M_{jj'}$ (with $j, j' = 1, 2$) are some real numbers and may be understood as elements of a 2×2 matrix M. Looking for an exponential solution of the type (80),

$$q_1 = c_1 e^{\lambda t}, \quad q_2 = c_2 e^{\lambda t}, \tag{4.88}$$

we get a more general system of two linear equations for the distribution coefficients

$$(M_{11} - \lambda)c_1 + M_{12}c_2 = 0, \qquad M_{21}c_1 + (M_{22} - \lambda)c_2 = 0. \tag{4.89}$$

These equations are consistent if

$$\begin{vmatrix} M_{11} - \lambda & M_{12} \\ M_{21} & M_{22} - \lambda \end{vmatrix} = 0, \tag{4.90}$$

giving us a quadratic characteristic equation

$$\lambda^2 - \lambda(M_{11} + M_{22}) + (M_{11}M_{22} - M_{12}M_{21}) = 0. \tag{4.91}$$

Its solution,[23]

$$\lambda_\pm = \frac{1}{2}(M_{11} + M_{22}) \pm \frac{1}{2}\left[(M_{11} - M_{22})^2 + 4M_{12}M_{21}\right]^{1/2}, \tag{4.92}$$

shows that the following situations are possible:

A. The expression under the square root, $(M_{11} - M_{22})^2 + 4M_{12}M_{21}$, is positive. In this case, both characteristic exponents $\lambda_\pm$ are real, and we can distinguish three sub-cases:

(i) Both λ_+ and λ_- are negative. In this case, the fixed point is evidently stable. Because of generally different magnitudes of exponents $\lambda_\pm$, the process presented on the *phase plane* $[q_1, q_2]$ (Fig. 8a) may be seen as consisting of two stages: first, a faster (with rate $|\lambda_-|$) relaxation to a linear *asymptote*,[24] and then a slower decline, with rate $|\lambda_+|$, along this line. Such fixed point is called the *stable node*.

(ii) Both λ_+ and λ_- are positive. This case (pretty rare in actual physical systems) of the *unstable node* differs from the previous one only by the direction of motion along the phase plane trajectories (see dashed arrows in Fig. 8a).

(iii) Finally, in the case of a *saddle* ($\lambda_+ > 0$, $\lambda_- < 0$) the system dynamics is different (Fig. 8b): after the rate-$|\lambda_-|$ relaxation to the λ_+-asymptote, the perturbation starts to grow, with rate λ_+, along one of two opposite directions. (The direction is determined on which side of another straight line, called *separatrix*, the system was initially.) It is evident that the saddle is an unstable fixed point.

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23 In terms of linear algebra, $\lambda_\pm$ are called the *eigenvalues*, and the corresponding sets $[C_1, C_2]_\pm$, the *eigenvectors* of matrix M.

24 The asymptote direction may be found by plugging the value of λ_+ back into Eq. (89) and finding the corresponding ratio $c_1/c_2 = x_1/x_2$.

B. The expression under the square root, $(M_{11}- M_{22})^2 + 4 M_{12}M_{21}$, is negative. In this case the square root in Eq. (92) is imaginary, with equal real parts of both roots, $\text{Re}\lambda_\pm = (M_{11} + M_{22})/2$, and opposite signs of their imaginary parts. As a result, here there can be just two types of fixed points:

(i) S*table focus*, at $(M_{11} + M_{22}) < 0$. The phase plane trajectories are spirals going to the center (i.e. toward the fixed point) – see Fig. 8c with solid arrow.

(ii) *Unstable focus*, taking place at $(M_{11} + M_{22}) > 0$, differs from the stable one only by the direction of motion along the phase trajectories – here it is from the dashed arrow in Fig. 8c.

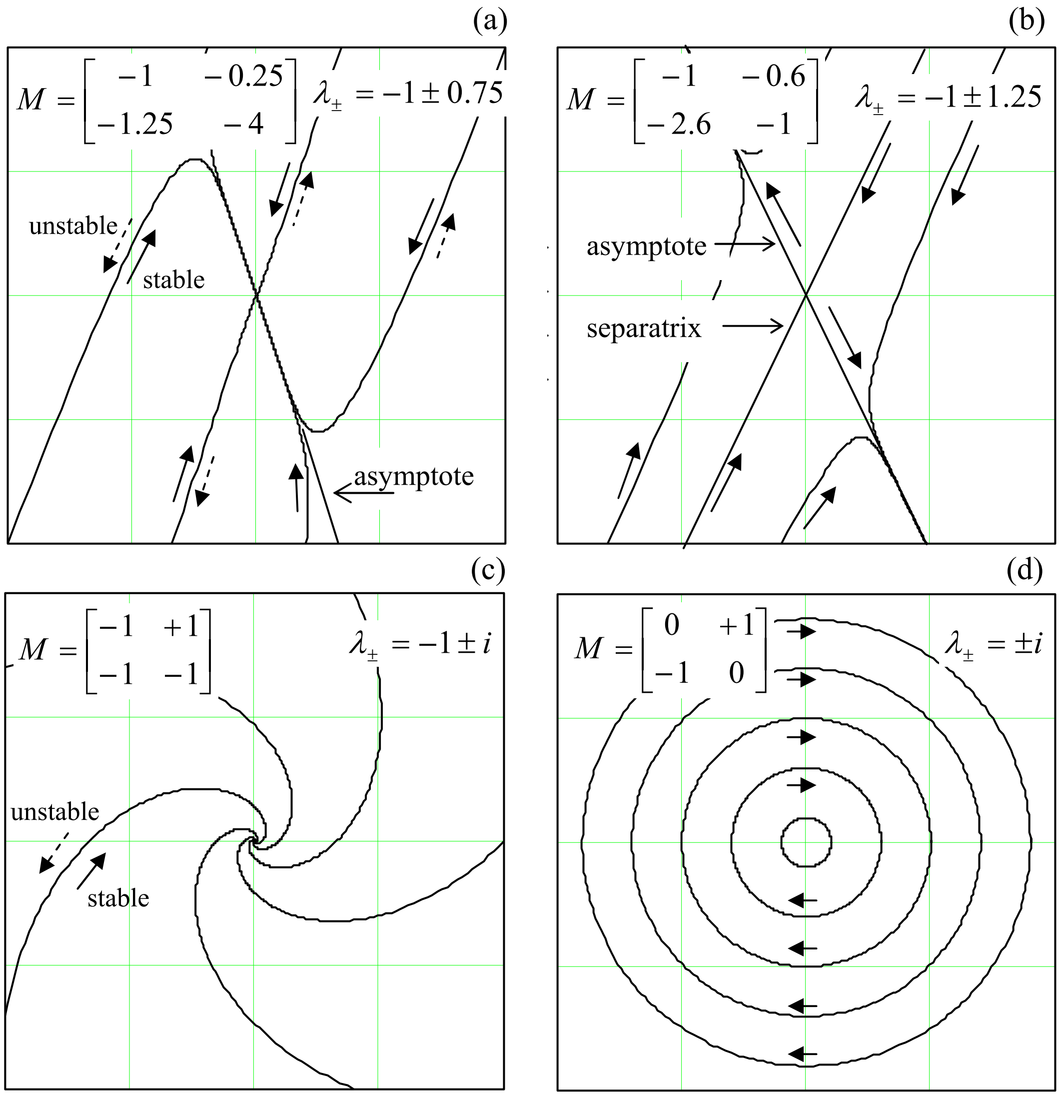

Fig. 4.8. Typical trajectories on the phase plane $[q_1, q_2]$ near fixed points of various types: (a) node, (b) saddle, (c) focus, and (d) center. The particular values of the matrix elements used the first three panels correspond to reduced equations (81) for parametric oscillators with $\xi = \delta$, and three different values of parameter $\mu\omega/4$: (a) 1.25, (b) 1.6 and (c) 0.

Sometimes the border case, $M_{11} + M_{22} = 0$, is also distinguished, and the corresponding fixed point is refereed to as the *center* (Fig. 8d). This classification makes sense because centers are typical for Hamiltonian systems whose first integral of motion may be frequently presented as the distance from a fixed point. For example, a harmonic oscillator without dissipation may be described by the system

$$\dot{q} = \frac{p}{m}, \quad \dot{p} = -m\omega_0^2 q, \tag{4.94}$$

which is evidently a particular case of Eq. (87) with $M_{11} = M_{22} = 0$, $M_{12}M_{21} = -\omega_0^2 < 0$, and hence $(M_{11}-M_{22})^2 + 4M_{12}M_{21} = -4\omega_0^2 < 0$, and $M_{11} + M_{22} = 0$. The phase plane of the system may be symmetrized by plotting q vs. the properly normalized momentum $\wp \equiv p/(m\omega_0)$. On the symmetrized plane, sinusoidal oscillations of various amplitudes A are presented circles of radius A about the center-type fixed point $A = 0$. In this case, the circular trajectories correspond to the conservation of the oscillator's energy

$$E = \frac{m\dot{q}^2}{2} + \frac{m\omega_0^2 q^2}{2} = \frac{m\omega_0^2}{2}\left[q^2 + \wp^2\right]. \tag{4.95}$$

This is a convenient moment for a brief discussion of the so-called *Poincaré* (or "*slow-variable*", or "*stroboscopic*") *plane*. From the point of view of the van der Pol method, sinusoidal oscillations $q(t) = A\cos(\omega t - \varphi)$, in particular those described by any circular trajectories on the real (or "fast") phase plane (Fig. 8c) correspond to a fixed point $\{A, \varphi\}$, which may conveniently presented by a steady geometrical point on a plane with such polar coordinates (Fig. 9a). (As follows from Eq. (4), the Cartesian coordinates on that plane are u and v.) It is evident that the quasi-sinusoidal process (41), with slowly changing A and φ, may be presented by its trajectory on this "Poincaré" plane. In many cases, such presentation is convenient than that on the "real" phase plane.

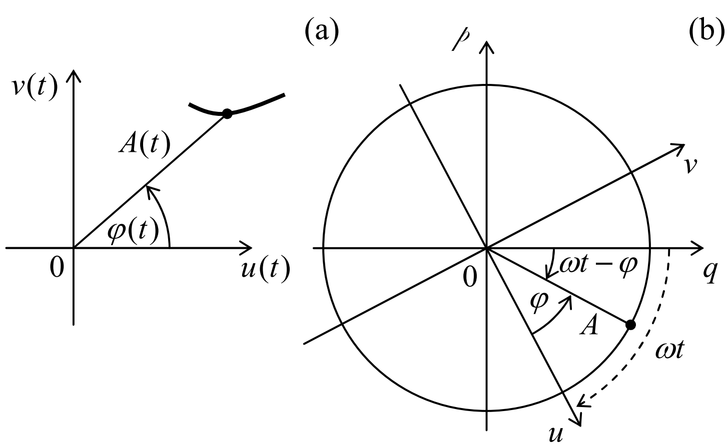

Fig. 4.9. (a) Presentation of a sinusoidal oscillation (point) and a slow transient (line) on the Poincaré plane, and (b) transfer from the "fast" phase plane to the "slow" (Poincaré) plane.

Figure 9b shows one possible way to visualize the relation between the "real" phase plane of an oscillator (in symmetrized variables q and $\wp$) and the "slow" Poincaré plane: it is sufficient to rotate the reference frame of the latter plane clockwise about the center, with angular velocity ω.[25] Another, "stroboscopic" way is to have a fast glance at the "fast" phase plane just once during the oscillation period $T = 2\pi/\omega$.

[25] This notion of phase plane rotation is essentially the basis for the popular modern name of the van der Pol method, the "rotating wave approximation" (RWA).

For example, let us return to the parametric excitation of oscillations, discussed in the last section. Figure 10 shows the classification of the trivial fixed point of a parametric oscillator, which follows from Eq. (83). As the parameter modulation depth μ is increased, the type of the trivial fixed point $A_1 = 0$ on the Poincaré plane changes from a stable focus (typical for a simple oscillator with damping) to a stable node and then to a saddle describing the parametric excitation. In the last case, the two directions of the perturbation growth, prominently featured in Fig. 8b, correspond to the two possible values of the oscillation phase φ mentioned in the end of Sec. 5.

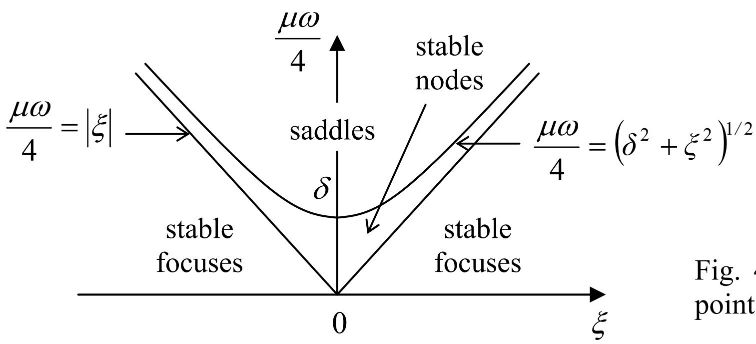

Fig. 4.10. Types of the trivial fixed point of a parametric oscillator.

4.7. Numerical approach

If the amplitude of oscillations, by whatever reason, becomes so large that the nonlinear terms in the equation describing a system are comparable to its linear terms, numerical methods are virtually the only study avenue available. In Hamiltonian 1D systems, such methods may be applied directly to integral (3.26), but dissipative and/or parametric systems typically lack first integrals of motion similar to Eq. (3.24), so that the initial differential equation has to be solved.

Let us discuss the general idea of such methods on the example of what mathematicians call the "*Cauchy problem*" (finding the solution for all moments of time, starting from known initial conditions) for first-order differential equation

$$\dot{q} = f(t, q). \tag{4.96}$$

(The generalization to a set of several such equations is straightforward.) Breaking the time axis into small, equal steps h (Fig. 9) we can reduce the equation integration problem to finding the function value in the next time point, $q_{n+1} \equiv q(t_{n+1}) = q(t_n + h)$ from the previously found value $q_n = q(t_n)$ (and, if necessary the values of q at other previous time steps). In the generic approach (called the *Euler method*), q_{n+1} is found using the following approximate formula:

$$\begin{aligned} q_{n+1} &= q_n + k, \\ k &\equiv h\, f(t_n, q_n). \end{aligned} \tag{4.97}$$

It is evident that this approximation is equivalent to the replacement of the genuine function $q(t)$, on the segment $[t_n, t_{n+1}]$, with the two first terms of its Taylor expansion in point t_n:

$$q(t_n + h) \approx q(t_n) + \dot{q}(t_n)h = q(t_n) + hf(t_n, x_n). \tag{4.98}$$

Such approximation has an error proportional to h^2. One could argue that making the step h sufficiently small the Euler's method error might be done arbitrary small, but even with the number-crunching power of modern computers the computation time necessary to reach sufficient accuracy may be too high for large problems. Besides that, at $h \to 0$ the rounding errors, unavoidable at any digital computing, may cause an increase, rather than the reduction of the overall error.

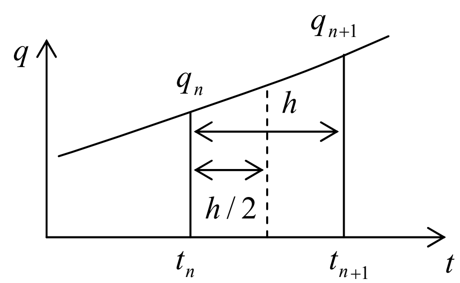

Fig. 4.11. The basic notions used at numerical integration of ordinary differential equations.

A more efficient way is to modify Eq. (97) to include the terms of the second order in h. There are several ways to do this, for example using the *2nd-order Runge-Kutta* method:

$$q_{n+1} = q_n + k_2,$$
$$k_2 \equiv h\,f(t_n + \frac{h}{2}, q_n + \frac{k_1}{2}), \quad k_1 \equiv h\,f(t_n, q_n). \tag{4.99}$$

One can readily check that this method gives the exact result is $q(t)$ is a quadratic parabola, and hence in the general case its errors are of the order of h^3. We see that the main idea here is to first break the segment $[t_n, t_{n+1}]$ in half (Fig. 11), then evaluate the RHP of the differential equation in the point intermediate (in both t and q) between points n and $(n + 1)$, and then use this information to predict q_{n+1}.

The advantage of the Runge-Kutta approach is that it can be readily extended to the 4th order, without an additional breaking of the interval $[t_n, t_{n+1}]$.:

$$q_{n+1} = q_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$
$$k_4 \equiv h\,f(t_n + h, q_n + k_3),\ k_3 \equiv h\,f(t_n + \frac{h}{2}, q_n + \frac{k_2}{2}),\ k_2 \equiv h\,f(t_n + \frac{h}{2}, q_n + \frac{k_1}{2}),\ k_1 \equiv h\,f(t_n, q_n). \tag{4.100}$$

This method reaches much lower error, $O(h^5)$, without being not too cumbersome. These features have made the 4th-order Runge-Kutta the default method for most numerical libraries. Its extension to higher orders is possible but requires more complex formulas and is justified only for some special cases, e.g., very abrupt functions $q(t)$.[26] The most frequent enhancement of the method is the automatic adjustment (reduction) of step h until the necessary accuracy (say, that specified by the user) has been reached.

Figure 12 shows a typical example of application of that method to the very simple problem of a damped linear oscillator, for two values of fixed time step h (expressed in terms of the number N of such steps per oscillation period). Black lines connect the points obtained by the 4th-order Runge-Kutta

[26] The most popular approaches in such cases are the *Richardson extrapolation*, the *Bulirsch-Stoer algorithm*, and a set of "prediction-correction" techniques, e.g. the *Adams-Bashforth-Moulton method* – see the literature recommended in MA, Footnote 4.

method, while the points connected by green lines present the exact analytical solution (22). A few-percent errors start to appear only at as few as ~10 time steps per period, so that the method is indeed very efficient. I will illustrate the convenience and handicaps of the numerical approach to the solution of dynamics problems on the discussion of the following topic.

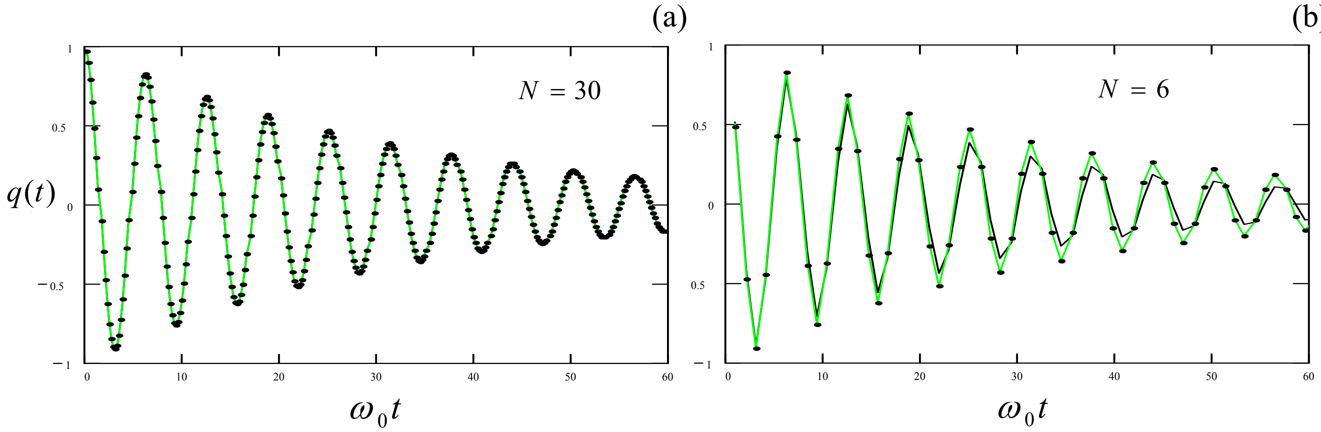

Fig. 4.12. Results of the fixed-point Runge-Kutta solution to the equation of linear oscillator with damping (with $\delta/\omega_0 = 0.03$) for: (a) 30 and (b) 6 points per oscillation period. The results are shown by points; lines are only the guide for the eye.

4.8. Harmonic and subharmonic oscillations

Figure 13 shows the numerically calculated[27] transient process and stationary oscillations in a linear oscillator and a very representative nonlinear system, the pendulum described by Eq. (42), both with the same resonance frequency ω_0 for small oscillations. Both systems are driven by a sinusoidal external force of the same amplitude and frequency (in this illustration, equal to the small-oscillation eigenfrequency ω_0 of both systems).

The plots show that despite a very substantial amplitude of the pendulum oscillations (an angle amplitude of about one radian) their waveform remains almost exactly sinusoidal. (In this particular case, the upper harmonic contents is about 0.5%, dominated by the 3^{rd} harmonic whose amplitude and phase are in a good agreement with Eq. (46).) On the other hand, the nonlinearity affects the oscillation amplitude and the transient law very substantially. These results show how effective are the small-parameter method and its van der Pol extension to transient processes, which have been discussed in Sections 2-5 above. Indeed, their validity range far exceeds what might be expected from the formal requirement $|q| << 1$.

27 All numerical results shown in this section have been obtained by the 4^{th}-order Runge-Kutta method with the automatic step adjustment which guarantees the relative accuracy of the order of 10^{-4}.

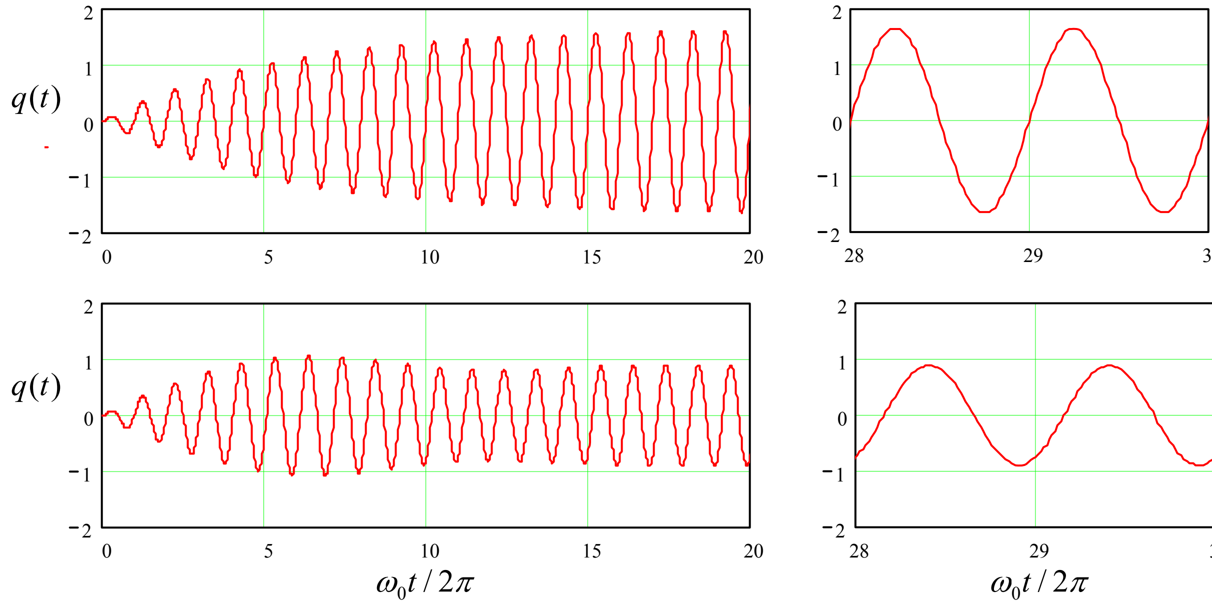

Fig. 4.13. Oscillations induced by a similar sinusoidal external force (turned on at $t = 0$) in two systems with the same small-oscillation frequency ω_0 and weak damping – a linear oscillator (two top panels) and a pendulum (two bottom panels). $\delta/\omega_0 = 0.03$, $f_0 = 0.1$, and $\omega = \omega_0$.

The upper harmonic content in the oscillation waveform may be sharply increased[28] by reducing the external force frequency to $\sim\omega_0/n$, where n is the number of the desirable harmonic. For example, Fig. 14a shows oscillations in a pendulum described by the same Eq. (42), but with frequency $\omega_0/3$.

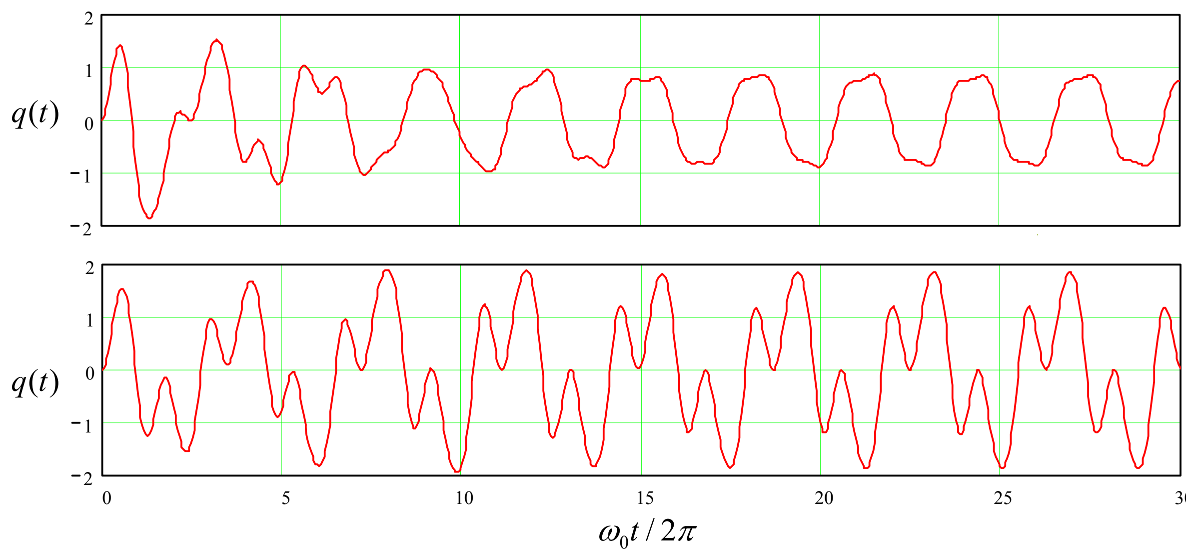

Fig. 4.14. Oscillations induced in a pendulum with damping $\delta/\omega_0 = 0.03$, driven by a sinusoidal external force of amplitude $f_0 = 0.75$, and frequency $\omega_0/3$ (top panel) and $(\omega_0/3)\times 0.8$ (bottom panel).

[28] This method is used in practice, for example, for the generation of EM oscillations in terahertz range (10^{12}-10^{13} Hz) which still lacks efficient electronic self-oscillators.

One can see that the 3rd harmonic amplitude may be even higher than that of the basic one, especially if the external frequency is additionally lowered (Fig. 14b) to accommodate for the deviation of the effective frequency $\omega_0(a)$ of own oscillations from its small-oscillation value ω_0 – see Eq. (49), Fig. 4 and their discussion in Sec. 2 above.

Note that effective generation of higher harmonics is only possible with adequate nonlinearity of the system. For example, consider the nonlinear terms like αq^3 explored in Secs. 2 and 3. If the waveform $q(t)$ is approximately sinusoidal, such terms can create only the basic and 3rd harmonics. The "pendulum nonlinearity" sin q cannot produce, in these conditions, any even (e.g., 2nd) harmonic. The most efficient generation of harmonics may be achieved using systems with the sharpest nonlinearities – e.g., semiconductor diodes whose current may follow an exponential dependence on the applied voltage through several orders of magnitude.

Generally, the higher harmonic generation by nonlinear systems might be readily anticipated. Indeed, the Fourier theorem tells us that any non-sinusoidal periodic function of time, e.g., an initially sinusoidal waveform distorted by nonlinearity, may be presented as a sum of its basic and higher harmonics with frequencies n times higher.

However, numerical modeling of nonlinear oscillators (as well as experiments with their physical implementations) bring more surprises. For example, Fig. 15 shows oscillations in a pendulum under effect of a strong sinusoidal force with a frequency close to $3\omega_0$. One can see that at some parameter values and initial conditions the system's oscillation spectrum is heavily contributed (almost dominated) by the 3rd *sub*harmonic, i.e. a component which is synchronous with the driving force but has the frequency three times lower, which is close to the own frequency of the system.

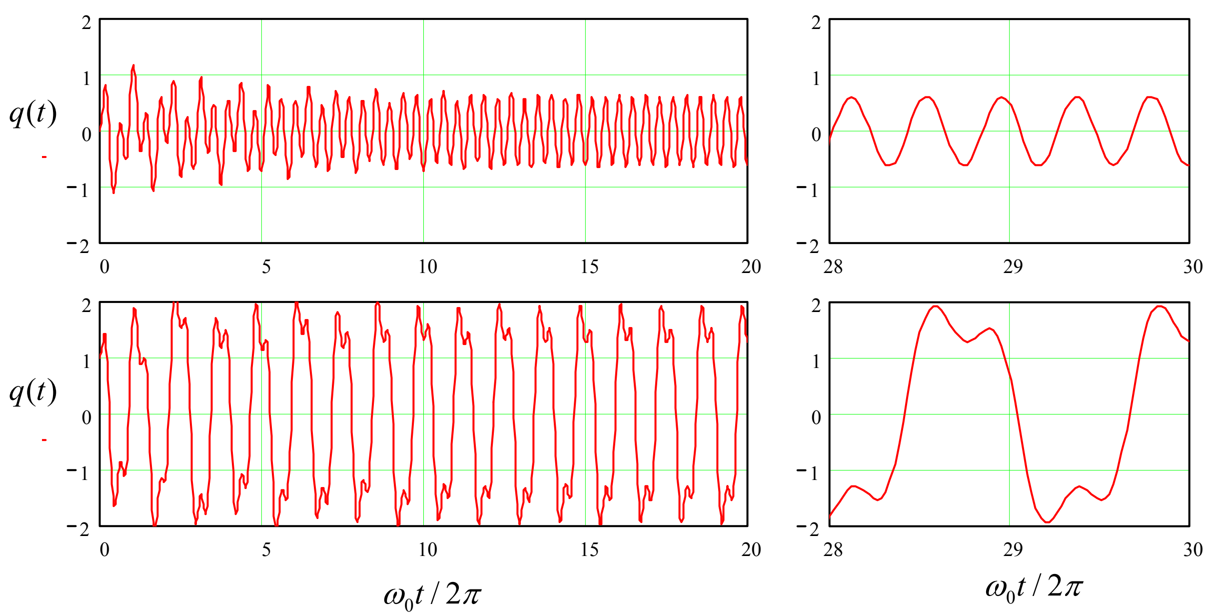

Fig. 4.15. Oscillations induced in a pendulum with δ/ω_0 = 0.03 by a sinusoidal external force of amplitude f_0 = 3 and frequency $3\omega_0 \times 0.8$, with initial conditions $q(0)$ = 0 (the top row) and $q(0)$ = 1 (the bottom row).

This counter-intuitive phenomenon may be explained as follows. If the subharmonic oscillations have appeared (together with the forced oscillations on the basic frequency):

$$q(t) \approx A\cos\Psi + A_{\text{sub}}\cos\Psi_{\text{sub}}, \tag{4.101a}$$

$$\Psi_{\text{sub}} \equiv \omega t - \varphi_{\text{sub}}, \quad \Psi \equiv 3\omega t - \varphi, \tag{4.101b}$$

the system's nonlinearity allows these oscillations to be self-sustained. Indeed, approximating sin q by the first nonlinear term αq^3 of its Taylor expansion, we have

$$\begin{aligned} q^3 &= (A_{\text{sub}}\cos\Psi_{\text{sub}} + A\cos\Psi)^3 \\ &= A_{\text{sub}}^3\cos^3\Psi_{\text{sub}} + 3A_{\text{sub}}^2 A\cos^2\Psi_{\text{sub}}\cos\Psi + 3A_{\text{sub}}A^2\cos\Psi_{\text{sub}}\cos^2\Psi + A^3\cos^3\Psi. \end{aligned} \tag{4.102}$$

While the first and the last terms of this expression depend only of amplitudes of the individual components of oscillations, the two middle terms are more interesting because they produce *combinational frequencies* of the two components. For our case, the second term,

$$3A_{\text{sub}}^2 A\cos^2\Psi_{\text{sub}}\cos\Psi = \frac{3}{4}A_{\text{sub}}^2 A\cos(\Psi - 2\Psi_{\text{sub}}) + ..., \tag{4.103}$$

of a special importance, because it produces, besides other combinational frequencies, the subharmonic component with the total phase

$$\Psi - 2\Psi_{\text{sub}} = \omega t - \varphi + 2\varphi_{\text{sub}}. \tag{4.104}$$

Thus, within a certain range of the mutual phase shift of the Fourier components, this nonlinear contribution can deliver energy from the external force to the subharmonic component, and can sustain it in the system.

A similar calculation for the second subharmonic shows that it can be sustained by even lower, quadratic nonlinearity of the system. This case has a special practical importance because the corresponding combinational term is linear in the subharmonic amplitude $A_{1/2}$, i.e. survives the equation linearization near the trivial fixed point. This mean that the second subharmonic may arise spontaneously, from infinitesimal fluctuations. (In contrast, the 3rd and higher subharmonics cannot be self-excited and always need an initial "kick-off" – compare two panels of Fig. 15.) Such excitation of the second subharmonic is very similar to the parametric excitation discussed in detail in Sec. 5, and this similarity is not coincidental.

Indeed, let us explore the quadratic nonlinearity γq^2 in a more general situation when the oscillations are a sum of the forced oscillations at the external force frequency 2ω, and an *arbitrary* weak perturbation:

$$\begin{aligned} q(t) &= A\cos(2\omega t - \varphi) + \tilde{q}(t), \quad |\tilde{q}| << A, \\ q^2 &\approx A^2\cos^2(2\omega t - \varphi) + 2\tilde{q}(t)A\cos(2\omega t - \varphi). \end{aligned} \tag{4.105}$$

The second term in the last formula is *exactly* similar to the term describing the parametric effects in Eq. (75). We may interpret this fact as follows: for a weak perturbation a system with a quadratic nonlinearity in the presence of a strong "*pumping*" signal of frequency ω is equivalent to a system with parameters changing in time with frequency ω. This fact is broadly used for the parametric excitation at high (e.g., optical) frequencies where the mechanical means of parameter change (see, e.g., Fig. 5) are not too practicable. The necessary quadratic nonlinearity at optical frequencies may be provided by several "nonlinear crystals", e.g., the lithium niobate.

Before finishing this chapter, let me elaborate a bit on a general topic: the relation between the numerical and analytical approaches to problems of dynamics (and physics as a whole). We have just seen that sometimes numerical solutions (like those shown in Fig. 15b) give vital clues on previously unanticipated phenomena such as the excitation of subharmonics ω/n with $n > 2$. (The turbulence and chaos phenomena, which will be discussed later in this course, present other examples of such "numerical discoveries".) One might also argue that in the absence of exact analytical solutions, numerical simulations may be the main theoretical tool for the study of such phenomena. These hopes are, however, muted by the problem which is frequently called the "*curse of dimensionality*", in which the last word refers to the number of input parameters of the problem to be solved.[29]

Indeed, let us have another look at Fig. 15. OK, we have been lucky to find a new phenomenon, the 3rd subharmonic generation, for a particular set of parameters (in that case, five of them: $\delta/\omega_0 = 0.03$, $3\omega/\omega_0 = 2.4$, $f_0 = 3$, $q(0) = 1$, $dq/dt\,(0) = 0$). Could we tell anything about how common this effect is? Are subharmonics with different n possible in the system? The only way to address these questions computationally is to carry out similar numerical simulations in many points of the d-dimensional (in this case, $d = 5$) space of parameters. Say, we have decided that breaking the reasonable range of each parameter to $N = 100$ points is sufficient. (For many problems, it is not – see, e.g., Sec. 9.1.) Then the total number of numerical experiments to carry out is $N^d = 10^{10}$ – not a simple task even for the powerful modern computing facilities. (Besides the pure CPU time expenditures, think how you would store and analyze the resulting data.) For many important problems of nonlinear dynamics, e.g., turbulence, the parameter dimensionality d is substantially larger, and the computer resources necessary for one numerical experiment, are much greater.

In the view of the curse of dimensionality concerns, approximate analytical considerations, like those outlined above for the subharmonic excitation, are absolutely invaluable. More generally, physics used to stand on two legs, experiment and (analytical) theory. The enormous progress of computer performance during a few last decades has provided it with one more point of support (a tail? :-) – numerical simulation. This does not mean we can afford to cut and throw away any of the legs.

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[29] In EM Sec. 1.2, I discuss the "curse" implications for a different case, when both analytical and numerical solutions to the same problem are possible.