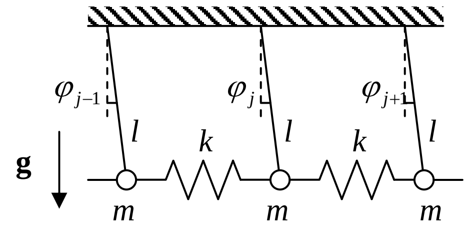
PHY 501 **Classical Mechanics and Dynamics** Fall 2009

Final exam with solutions

Problem F.1 (300 points). Calculate the dispersion relation $\omega(q)$ for small longitudinal 1D waves in the system of similar, spring-coupled pendula (see Fig. on the right). Sketch the relation, and list its key properties.



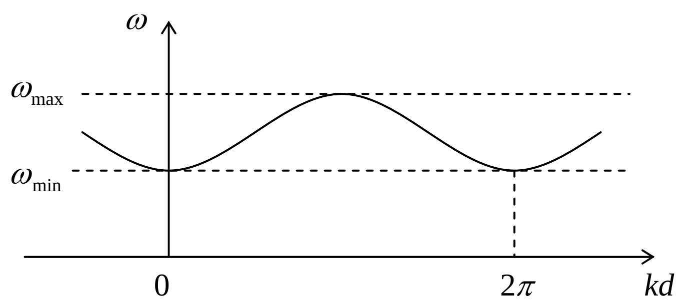
Solution: The linearized equation of motion of the *j*-th pendulum is

$$\ddot{\varphi}_j = -\Omega^2\varphi_j + \omega_0^2(\varphi_{j-1} - \varphi_j) + \omega_0^2(\varphi_{j+1} - \varphi_j),$$

where $\Omega \equiv (g/l)^{1/2}$ and $\omega_0 \equiv (k/m)^{1/2}$. Looking for a particular solution of this equation in the form of a traveling wave, $\varphi_j = a \exp\{i(kx_j - \omega t)\}$, with $x_j = jd$, where d is the structure period, we get the dispersion relation

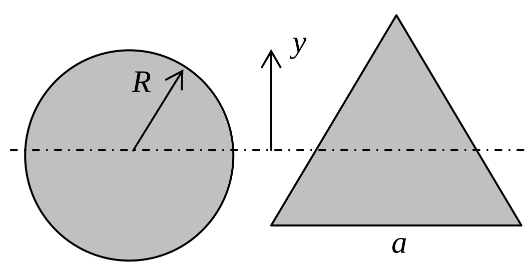
$$\omega = \left(4\omega_0^2 \sin^2\frac{kd}{2} + \Omega^2\right)^{1/2}.$$

This relation is sketched below. It shows that the system may sustain waves of any frequency between $\omega_{\min} = \Omega$ and $\omega_{\max} = [4\omega_0^2 + \Omega^2]^{1/2}$.



Notice the absence of acoustic branches. This is typical for systems of particles bound to the vicinity of their equilibrium positions in the "absolute" (lab) reference frame.

Problem F.2 (400 points). Two thin rods of the same length and mass have been made of the same elastic, isotropic material. The cross-section of one of them is a circle, while another one is an equilateral triangle (see Fig. on the right). Which of the rods is more stiff for bending along its length? Quantify the relation. Does the result depend on the bending plane orientation?



Solution: According to the analysis of Sec. 6.5, the rod stiffness may be characterized by its curvature radius which is in turn proportional to the "moment of inertia"

$$I = \int_A y^2 dA,$$

where y is the distance of the point from the neutral plane which passes through the cross-section's center of mass – see the dashed line in the Fig. above. For the circle of radius R,

$$I_C = \int_0^R r dr \int_0^{2\pi} d\varphi\, y^2 = \int_0^R r dr \int_0^{2\pi} d\varphi (r \sin\varphi)^2 = \int_0^R r^3 dr \int_0^{2\pi} \sin^2\varphi\, d\varphi = \pi \int_0^R r^3 dr = \frac{\pi}{4} R^4,$$

and is of course independent of the bending direction. For an equilateral triangle with side a, the moment of inertia has been calculated in Homework Problem 8.1:

$$I_T = \frac{\sqrt{3}}{96} a^4,$$

and also does not depend on the bending plane orientation. In order to have the same rod mass using the same material, the areas of these two cross-sections have to be equal:

$$\pi R^2 = \frac{\sqrt{3}}{4} a^2.$$

From here,

$$\frac{I_C}{I_T} = \frac{(\pi/4)R^4}{(\sqrt{3}/96)a^4} = \frac{\pi}{4}\frac{96}{\sqrt{3}}\left(\frac{\sqrt{3}}{4\pi}\right)^2 = \frac{9}{2\pi\sqrt{3}} \approx 0.82.$$

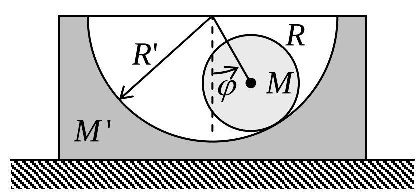
Hence, the round cross-section gives a slightly lower bending stiffness than the triangular cross-section.

Problem F.3 (450 points). A solid, uniform, round cylinder of mass M can roll, without slipping, over a concave, round cylindrical surface of a block of mass M', in Earth's gravity field – see Fig. on the right. The block can slide without friction on a horizontal surface. Using the Lagrangian formalism,

(i) find the frequency of small oscillations of the system near the equilibrium, and
(ii) sketch the oscillation mode for the particular case $M' = M$, $R' = 2R$.

Solution: The system's Lagrangian may be presented as

$$L = K - U = \frac{M'}{2}\dot{X}'^2 + \frac{M}{2}\dot{X}^2 + \frac{M}{2}\dot{Y}^2 + \frac{I}{2}\omega^2 - MgY,$$



where X is the (horizontal) coordinate of the block, while X' and Y' are the Cartesian coordinates the cylinder's center of mass (all in the lab reference frame), ω is the cylinder's angular velocity, and I is its moment of inertia for rotation about the center of mass. Using X' and angle φ (defined as shown in Fig. above) as generalized coordinates, we may use relation (7.48) of the lecture notes to write

$$\omega = -\frac{R' - R}{R}\dot{\varphi},$$

and rewrite L as

$$L=\frac{M'}{2}\dot{X}'^2+\frac{M}{2}\left[\dot{X}'+(R'-R)\dot{\varphi}\cos\varphi\right]^2+\frac{M}{2}\left[(R'-R)\dot{\varphi}\sin\varphi\right]^2+\frac{I}{2}\left[\frac{R'-R}{R}\dot{\varphi}\right]^2+Mg(R'-R)\cos\varphi\,.$$

The standard procedure of differentiation of L, together the well-known formula $I = MR^2/2$, gives the following Lagrangian equations of motion:

$$\frac{d}{dt}\left[(M+M')\dot{X}'+M(R'-R)\dot{\varphi}\cos\varphi\right]=0,$$
$$\frac{d}{dt}\left[M(R'-R)\dot{X}'\cos\varphi+\frac{3}{2}M(R'-R)^2\dot{\varphi}\right]+M(R'-R)\dot{X}'\dot{\varphi}\sin\varphi+Mg(R'-R)\sin\varphi=0.$$

It is useful to notice that the first of these equations immediately yields a first integral of motion,[1]

$$P_{X'}\equiv\frac{\partial L}{\partial\dot{X}'}=(M+M')\dot{X}'+M(R'-R)\dot{\varphi}\cos\varphi\;=\text{const}\,.$$

Linearizing the equation of motion for small deviations from equilibrium, we get

$$(M+M')\ddot{X}'+M(R'-R)\ddot{\varphi}=0,$$
$$\ddot{X}'+(3/2)(R'-R)\ddot{\varphi}+g\varphi=0.$$

Looking for the solution of this system of linear, homogeneous differential equations in the standard form

$$\varphi(t)=Ae^{-i\omega t},\quad X'(t)=A'e^{-i\omega t},$$

we get two linear algebraic equations for amplitudes A and A':

$$(M+M')A'+M(R'-R)A=0,$$
$$-\omega^2\left[A'+(3/2)(R'-R)A\right]+gA=0.$$

The simplest way to deal with this system is to express one of coordinates, e.g., A', from the first of these equations,

$$A'=-A\frac{(R'-R)M}{M+M'},\qquad (*)$$

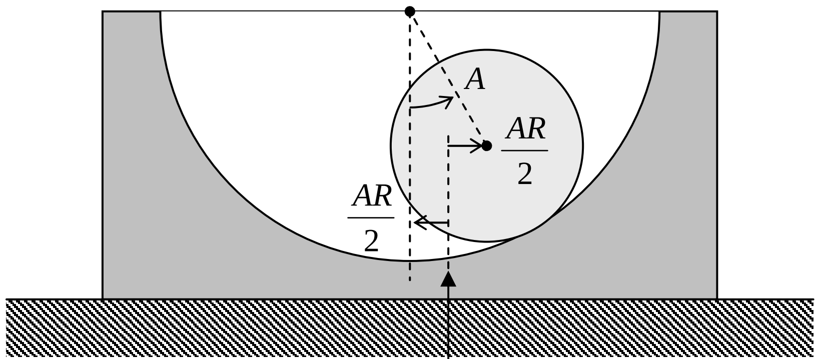
and exclude it from the second equation, so that we immediately get the final result which may be presented as

$$\omega^2=\frac{g}{l_{\text{ef}}},\quad l_{\text{ef}}\equiv(R'-R)\left(\frac{3}{2}-\frac{M}{M+M'}\right)>0.$$

Generally, the eigenfrequency values have to be plugged back into the equations relating oscillation amplitudes. However, for our simple system which (due to the conservation of $P_{X'}$) has just one eigenfrequency, the amplitudes are connected by Eq. (*) which does not depend on ω. For the

[1] This fact could be immediately noticed from the Lagrangian, because it does not depend explicitly on coordinate X', so that the corresponding generalized momentum $P_{X'}$ is conserved. Physically, this is just the x-component of the total linear momentum of the system, which is conserved because both external forces acting on the system, gravity and support reaction, are vertical. Another first integral of motion is energy $E = K + U$.

particular case $M' = M$, $R' = 2R$, this relation yields $A' = - AR/2$. This oscillation mode is sketched in Fig. below.

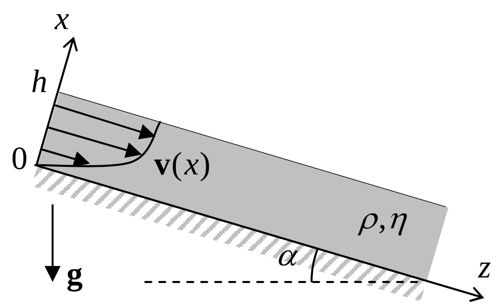


Notice also that according to Eq. (*), the relation between amplitudes of small oscillations of horizontal coordinates of the block (X') and the cylinder ($X \approx X' + (R' - R)\varphi$) in the lab frame,

$$A_X = A' + (R' - R)A = -\frac{M'}{M}A',$$

does not depend on ratio R/R'. In our particular case ($M' = M$), $A_X = - A' = -AR/2$ – see Fig. above.

Problem F.4 (450 points). A layer, of thickness h, of a heavy, viscous, incompressible fluid flows down a long and wide incline plane, under its own weight – see Fig. on the right. Find the stationary velocity distribution profile, and the total fluid discharge (per unit width.)



Solution: With the coordinate choice shown in Fig. on the right, $\mathbf{v} = \mathbf{n}_z v(x)$, so that the z-th component of the Navier-Stokes equation (8.44), in the stationary regime, becomes

$$\rho g \sin\alpha + \eta \frac{d^2 v}{dx^2} = 0.$$

(Due to the open surface of the fluid, it cannot sustain any pressure gradient in z direction.) This equation should be solved with boundary conditions

$$v(0) = 0, \quad \frac{dv}{dx}(h) = 0,$$

the latter one resulting from the absence of force in z direction (i.e. the stress component σ_{zx}) on the surface. An elementary integration yields

$$v(x) = \frac{\rho g \sin\alpha}{\eta}\frac{x(2h - x)}{2}, \quad \frac{Q}{w} \equiv \rho\int_0^h v(x)dx = \frac{\rho g h^3 \sin\alpha}{3}.$$