

## Selected Mathematical Formulas

which are used in this lecture course series, but not always remembered by students (and instructors :-)

### 1. Constants

- Euclidean circle's length-to-diameter ratio

$$\pi = 3.141\,592\,653\dots; \quad \sqrt{\pi} \approx 1.77. \quad (1.1)$$

- Natural logarithm base:

$$e \equiv \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.718\,281\,828\dots; \quad (1.2a)$$

from that number, the logarithm base conversion factors are as follows:

$$\frac{\ln x}{\log_{10} x} = \ln 10 \approx 2.303, \quad \frac{\log_{10} x}{\ln x} = \frac{1}{\ln 10} \approx 0.434. \quad (1.2b)$$

- The Euler (or "Euler-Mascheroni") constant (for the definition, see EM Sec. 2.4) :

$$\gamma = 0.57721566490\dots; \quad e^\gamma \approx 1.781. \quad (1.3)$$

### 2. Combinatorics, sums, and series

#### (i) Combinatorics

- The number of different *permutations*, i.e. *ordered* sequences of  $k$  elements selected from a set of  $n$  distinct elements, is

$${}^n P_k \equiv n \cdot (n-1) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}; \quad (2.1a)$$

in particular, the number of different permutations of *all* elements of the set ( $n = k$ ) is

$${}^k P_k = k \cdot (k-1) \cdot \dots \cdot 2 \cdot 1 = k!. \quad (2.1b)$$

- The number of different *combinations*, i.e. *unordered* sequences of  $k$  elements from a set of  $n \geq k$  distinct elements, is equal to the "binomial coefficient"

$${}^n C_k \equiv \binom{n}{k} \equiv \frac{{}^n P_k}{{}^k P_k} = \frac{n!}{k!(n-k)!}, \quad (2.2)$$

In an alternative, very popular "ball/box language",  ${}^n C_k$  is the number of different ways to put in a box, in an arbitrary order,  $k$  balls from the total number of  $n$ , while considering each ball distinct from others.

- A generalization of the binomial coefficient notion is the "multinomial coefficient",

$${}^n C_{k_1, k_2, \dots, k_l} \equiv \frac{n!}{k_1! k_2! \dots k_l!}, \quad \text{with } n = k_1 + k_2 + \dots + k_l, \quad (2.3)$$

which, in the standard mathematical language, is a number of different permutations in a multiset of  $l$  distinct element types from an  $n$ -element set which contains  $k_j$  ( $j = 1, 2, \dots, l$ ) elements of each type. In the “ball/box language”, coefficient (2.3) is the number of different ways to distribute  $n$  balls between  $l$  different boxes, each time keeping the number ( $k_j$ ) of balls in the  $j$ -th box fixed, but ignoring their order in the box. The binomial coefficient  ${}^n C_k$ , defined by Eq. (2.2), is evidently a particular case of the multinomial coefficient for  $l = 2$ , so that if  $k_1 \equiv k$ , then  $k_2 = n - k$ .

(ii) Sums and series

- Arithmetic progression:

$$r + 2r + \dots + nr \equiv \sum_{k=1}^n kr = \frac{n(r + nr)}{2}; \quad (2.4a)$$

in particular, at  $r = 1$  it is reduced to the sum of  $n$  first natural numbers:

$$1 + 2 + \dots + n \equiv \sum_{k=1}^n k = \frac{n(n+1)}{2}. \quad (2.4b)$$

- Sum of squares of  $n$  first natural numbers:

$$1^2 + 2^2 + \dots + n^2 \equiv \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}. \quad (2.5)$$

- The *Riemann zeta function*:

$$\zeta(s) \equiv 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \sum_{k=1}^{\infty} \frac{1}{k^s}; \quad (2.6a)$$

the particular values frequently met in applications are

$$\zeta\left(\frac{3}{2}\right) \approx 2.612, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta\left(\frac{5}{2}\right) \approx 1.341, \quad \zeta(3) \approx 1.202, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(5) \approx 1.037. \quad (2.6b)$$

- Finite geometric progression (for real  $\lambda \neq 1$ ):

$$1 + \lambda + \lambda^2 + \dots + \lambda^{n-1} \equiv \sum_{k=0}^{n-1} \lambda^k = \frac{1 - \lambda^n}{1 - \lambda}; \quad (2.7a)$$

in particular, if  $\lambda^2 < 1$ , the progression has a finite limit at  $n \rightarrow \infty$  (the “geometric series”):

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \lambda^k = \sum_{k=0}^{\infty} \lambda^k = \frac{1}{1 - \lambda}. \quad (2.7b)$$

- Binomial sum:

$$(1+a)^n = \sum_{k=0}^n {}^n C_k a^k, \quad (2.8)$$

where  ${}^n C_k$  are the binomial coefficients defined by Eq. (2.2).

- The Stirling formula:

$$\lim_{n \rightarrow \infty} \ln(n!) = n(\ln n - 1) + \frac{1}{2} \ln(2\pi n) + \frac{1}{12n} - \frac{1}{360n^3} + \dots; \quad (2.9)$$

for most applications in physics, the first term (first derived by A. de Moivre) is sufficient.

- The Taylor (or “Taylor-Maclaurin”) series: for any infinitely differentiable function  $f(x)$ :

$$\lim_{\tilde{x} \rightarrow 0} f(x_0 + \tilde{x}) = f(x) + \frac{df(x)}{dx} \Big|_{x=x_0} \tilde{x} + \frac{1}{2!} \frac{d^2 f(x)}{dx^2} \Big|_{x=x_0} \tilde{x}^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f(x)}{dx^k} \Big|_{x=x_0} \tilde{x}^k; \quad (2.10)$$

note that for many functions this series converges only within a limited, sometimes small range of deviations  $\tilde{x}$ .

- The Euler-Maclaurin formula (valid for any infinitely differentiable function  $f$ ):

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x) dx - \frac{1}{2} f(0) - \frac{1}{6} \cdot \frac{1}{2!} \frac{df(x)}{dx} \Big|_{x=0} + \frac{1}{30} \cdot \frac{1}{4!} \frac{d^3 f(x)}{dx^3} \Big|_{x=0} + \dots; \quad (2.11)$$

the coefficients participating in this formula are the so-called *Bernoulli numbers*:

$$B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = \frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = \frac{1}{30}, \dots \quad (2.12)$$

Please notice that, unfortunately, definitions of  $B_k$  (or rather their signs and indices) differ even in the most popular handbooks. Fortunately, this nomenclature controversy does not affect Eq. (2.11).

### 3. Trigonometric functions

- Sums of two functions of arbitrary arguments:

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{B-A}{2}, \quad (3.1a)$$

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}, \quad (3.1b)$$

$$\sin A \pm \sin B = 2 \sin \frac{A \pm B}{2} \cos \frac{\pm B - A}{2}. \quad (3.1c)$$

- Products:

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B), \quad (3.2a)$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B), \quad (3.2b)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B); \quad (3.2c)$$

for the particular case of equal arguments,  $B = A$ , these formulas yield expressions for squares of trigonometric functions, and their product:

$$\cos^2 A = \frac{1}{2}(\cos 2A + 1), \quad \sin^2 A = \frac{1}{2}(1 - \cos 2A), \quad \sin A \cos A = \frac{1}{2} \sin 2A. \quad (3.2d)$$

- Cubes of trigonometric functions:

$$\cos^3 A = \frac{3}{4} \cos A + \frac{1}{4} \cos 3A, \quad \sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A. \quad (3.3)$$

#### 4. General differentiation

- Full differential of a product of two functions:

$$d(fg) = (df)g + f(dg). \quad (4.1)$$

- Full differential of a function of several independent arguments,  $f(x_1, x_2, \dots, x_n)$ :

$$df = \sum_{k=1}^n \frac{\partial f}{\partial x_k} dx_k. \quad (4.2)$$

#### 5. General integration

- Integration by parts (immediately follows from Eq. (4.1)):

$$\int_{g(A)}^{g(B)} f dg = fg \Big|_A^B - \int_{f(A)}^{f(B)} g df. \quad (5.1)$$

- Numerical (approximate) integration of 1D functions: the simplest “trapezoidal rule”,

$$\int_a^b f(x) dx \approx h \left[ f\left(a + \frac{h}{2}\right) + f\left(a + \frac{3h}{2}\right) + \dots + f\left(b - \frac{h}{2}\right) \right] = h \sum_{n=1}^N f\left(a - \frac{h}{2} + nh\right), \quad h \equiv \frac{b-a}{N}. \quad (5.2)$$

has relatively low accuracy (error of the order of  $(h^3/12)d^2f/dx^2$  per step), so that the following “Simpson formula”,

$$\int_a^b f(x) dx \approx \frac{h}{3} [f(a) + 4f(a+h) + 2f(a+2h) + \dots + 4f(b-h) + f(b)], \quad h \equiv \frac{b-a}{2N}, \quad (5.3)$$

whose error per step scales as  $(h^5/180)d^4f/dx^4$ , is used much more frequently.<sup>1</sup>

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<sup>1</sup> Higher-order formulas (e.g., the “Bode rule”), and other guidance including ready-for-use codes for computer calculations may be found, for example, in the popular reference texts by W. H. Press *et al.*, cited in the References section below. Besides that, some advanced codes are used as subroutines in the software packages listed in the same section. In some cases, the Euler-Maclaurin formula (2.11) also may be useful for numerical integration.

## 6. A few 1D integrals of elementary functions<sup>2</sup>

### (i) Indefinite integrals

- Integrals with  $(1 + \xi^2)^{1/2}$ :

$$\int (1 + \xi^2)^{1/2} d\xi = \frac{\xi}{2} (1 + \xi^2)^{1/2} + \frac{1}{2} \ln \left| \xi + (1 + \xi^2)^{1/2} \right|, \quad (6.1)$$

$$\int \frac{d\xi}{(1 + \xi^2)^{1/2}} = \ln \left| \xi + (1 + \xi^2)^{1/2} \right|. \quad (6.2)$$

- Integrals with  $(\xi^2 + 2a\xi - 1)^{1/2}$ :

$$\int \frac{d\xi}{\xi(\xi^2 + 2a\xi - 1)^{1/2}} = \arccos \frac{a\xi - 1}{|\xi|(a^2 + 1)^{1/2}}. \quad (6.3)$$

### (ii) Semi-definite integrals:

- Integrals with  $1/(e^\xi \pm 1)$ :

$$\int_a^\infty \frac{d\xi}{e^\xi + 1} = \ln(1 + e^{-a}), \quad (6.4a)$$

$$\int_{a>0}^\infty \frac{d\xi}{e^\xi - 1} = \ln \frac{1}{1 - e^{-a}}. \quad (6.4b)$$

### (iii) Definite integrals

- Integrals with  $e^{-\xi}$ :

$$\int_0^\infty \xi^{s-1} e^{-\xi} d\xi = \Gamma(s), \quad (6.5a)$$

where  $\Gamma(s)$  is the *gamma-function* whose its main property is

$$\Gamma(n) = (n-1)!, \quad \text{for } n = 1, 2, \dots; \quad (6.5b)$$

other particularly important values of the gamma-function are

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{1 \cdot 3}{2 \cdot 2}\sqrt{\pi}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2}\sqrt{\pi}, \dots \quad (6.5c)$$

- Integrals with  $(1 - \xi^{2n})^{1/2}$ :

$$\int_0^1 \frac{d\xi}{[1 - \xi^{2n}]^{1/2}} = \frac{\sqrt{\pi}}{2n} \Gamma\left(\frac{1}{2n}\right) / \Gamma\left(\frac{n+1}{2n}\right). \quad (6.6)$$

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<sup>2</sup> A powerful (and free :-)) interactive online tool for working out indefinite 1D integrals is available at <http://integrals.wolfram.com/index.jsp>.

- Integrals with  $1/(e^\xi \pm 1)$ :

$$\int_0^\infty \frac{\xi^{s-1} d\xi}{e^\xi + 1} = (1 - 2^{1-s}) \Gamma(s) \zeta(s), \quad \text{for } s > 0, \quad (6.7a)$$

$$\int_0^\infty \frac{\xi^{s-1} d\xi}{e^\xi - 1} = \Gamma(s) \zeta(s), \quad \text{for } s > 1, \quad (6.7b)$$

where  $\zeta(s)$  is the Riemann zeta-function – see Eq. (2.6). For the particular case  $s = 1$  (when Eq. (6.7a) yields uncertainty),

$$\int_0^\infty \frac{d\xi}{e^\xi + 1} = \ln 2. \quad (6.7c)$$

Particular cases: for  $s = 2n$ ,

$$\int_0^\infty \frac{\xi^{2n-1} d\xi}{e^\xi + 1} = \frac{2^{2n-1} - 1}{2n} \pi^{2n} B_{2n}, \quad (6.7d)$$

$$\int_0^\infty \frac{\xi^{2n-1} d\xi}{e^\xi - 1} = \frac{(2\pi)^{2n}}{4n} B_{2n}. \quad (6.7e)$$

where  $B_n$  are the Bernoulli numbers – see Eq. (2.12).

- Integrals with  $\exp\{-\xi^2\}$ :<sup>3</sup>

$$\int_0^\infty \xi^s e^{-\xi^2} d\xi = \frac{1}{2} \Gamma\left(\frac{s+1}{2}\right), \quad \text{for } s > -1. \quad (6.8)$$

## 7. 3D vector products

(i) Definitions:

- Scalar product:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^3 a_j b_j, \quad (7.1)$$

where  $a_j$  and  $b_j$  are vector components in any orthogonal coordinate system. In particular, vector squared (the same as norm squared):

$$a^2 \equiv \mathbf{a} \cdot \mathbf{a} = \sum_{j=1}^3 a_j^2 \equiv \|\mathbf{a}\|^2. \quad (7.2)$$

- Vector (“cross-”) product:

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<sup>3</sup> This formula follows directly from definition (6.5a) of the gamma-function, after substitution  $\xi \rightarrow \xi^2$ .

$$\mathbf{a} \times \mathbf{b} \equiv \mathbf{n}_1(a_2b_3 - a_3b_2) + \mathbf{n}_2(a_3b_1 - a_1b_3) + \mathbf{n}_3(a_1b_2 - a_2b_1) = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad (7.3)$$

where  $\{\mathbf{n}_j\}$  is the set of mutually perpendicular unit vectors<sup>4</sup> along the corresponding coordinate system directions.<sup>5</sup> In particular, Eq. (7.3) yields

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}. \quad (7.4)$$

(ii) Corollaries (readily verified by Cartesian components):

- Double vector product (the so-called “bac minus cab rule”):

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \quad (7.5)$$

- Mixed scalar-vector product (the “operator rotation rule”):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \quad (7.6)$$

- Scalar product of vector products:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}); \quad (7.7a)$$

in the particular case of two similar operands (say,  $\mathbf{a} = \mathbf{c}$  and  $\mathbf{b} = \mathbf{d}$ ), the last formula is reduced to

$$(\mathbf{a} \times \mathbf{b})^2 = (ab)^2 - (\mathbf{a} \cdot \mathbf{b})^2. \quad (7.7b)$$

## 8. Differentiation in 3D Cartesian coordinates

- Definition of the “del” (or “nabla”) vector-operator:<sup>6</sup>

$$\nabla \equiv \sum_{j=1}^3 \mathbf{n}_j \frac{\partial}{\partial r_j}, \quad (8.1)$$

where  $r_j$  is the linear (“Cartesian”) coordinate along direction  $\mathbf{n}_j$ . In accordance with this definition, del acting on a *scalar* function of coordinates,  $f(\mathbf{r})$ ,<sup>7</sup> gives its gradient:

$$\nabla f \equiv \sum_{j=1}^3 \mathbf{n}_j \frac{\partial f}{\partial r_j} \equiv \mathbf{grad} f, \quad (8.2)$$

i.e., a new *vector*.

- The “scalar product” of del by a *vector* function

<sup>4</sup> Popular alternative notations for this vector set are  $\{\mathbf{e}_j\}$  and  $\{\hat{\mathbf{r}}_j\}$ .

<sup>5</sup> It is easy to use Eq. (7.3) to check that the direction of the product vector corresponds to the “corkscrew rule”: if we rotate the first operand toward the second one, the usual corkscrew moves in the direction of the product.

<sup>6</sup> One can also meet the following notation:  $\nabla \equiv \partial / \partial \mathbf{r}$ , which may be convenient in some cases, but misleading in others, so it will not be used in these lecture notes.

<sup>7</sup> In this, and next 4 sections, all scalar and vector functions are assumed to be differentiable.

$$\mathbf{f}(\mathbf{r}) \equiv \sum_{j=1}^3 \mathbf{n}_j f_j(\mathbf{r}), \quad (8.3)$$

compiled formally following Eq. (7.1), is a *scalar* function – the “divergence” of the initial function:

$$\nabla \cdot \mathbf{f} \equiv \sum_{j=1}^3 \frac{\partial f_j}{\partial r_j} \equiv \mathbf{div} \mathbf{f}, \quad (8.4)$$

while the “vector product” of  $\nabla$  and  $\mathbf{f}$ , formed in a formal accordance with Eq. (7.3), is a new vector – the curl<sup>8</sup> of  $\mathbf{f}$ :

$$\nabla \times \mathbf{f} \equiv \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ \frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2} & \frac{\partial}{\partial r_3} \\ f_1 & f_2 & f_3 \end{vmatrix} = \mathbf{n}_1 \left( \frac{\partial f_3}{\partial r_2} - \frac{\partial f_2}{\partial r_3} \right) + \mathbf{n}_2 \left( \frac{\partial f_1}{\partial r_3} - \frac{\partial f_3}{\partial r_1} \right) + \mathbf{n}_3 \left( \frac{\partial f_2}{\partial r_1} - \frac{\partial f_1}{\partial r_2} \right) \equiv \mathbf{curl} \mathbf{f}. \quad (8.5)$$

- One more frequently met<sup>9</sup> “product” is  $(\mathbf{f} \cdot \nabla) \mathbf{g}$ , where  $\mathbf{f}$  and  $\mathbf{g}$  are two arbitrary vector functions of  $\mathbf{r}$ . This product should be also understood in the sense dictated by Eq. (7.1), i.e. as a vector whose  $j$ -th component is

$$[(\mathbf{f} \cdot \nabla) \mathbf{g}]_j = \sum_{j'=1}^3 f_{j'} \frac{\partial g_{j'}}{\partial r_{j'}}. \quad (8.5)$$

## 9. The Laplace operator

- Definition in Cartesian coordinates (in a formal accordance with Eq. (7.2)):

$$\nabla^2 \equiv \nabla \cdot \nabla = \sum_{j=1}^3 \frac{\partial^2}{\partial r_j^2}. \quad (9.1)$$

- According to the definition, the Laplace operator acting on a *scalar* function of coordinates gives a new scalar function:

$$\nabla^2 f \equiv \nabla \cdot (\nabla f) = \mathbf{div} (\mathbf{grad} f) = \sum_{j=1}^3 \frac{\partial^2 f}{\partial r_j^2}. \quad (9.2)$$

- On the other hand, acting on a *vector* function (8.3), operator  $\nabla^2$  returns another vector:

$$\nabla^2 \mathbf{f} = \sum_{j=1}^3 \mathbf{n}_j \nabla^2 f_j. \quad (9.3)$$

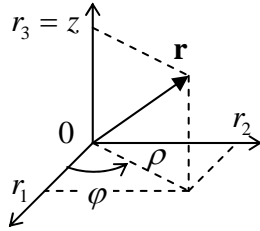
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<sup>8</sup> In the European tradition, this operator is called “rotor” and denoted as **rot**.

<sup>9</sup> See, e.g., Eqs. (11.5) and (11.6) below.

**10. Operators  $\nabla$  and  $\nabla^2$  in the most important systems of orthogonal coordinates<sup>10</sup>**

(i) Cylindrical<sup>11</sup> coordinates  $\{\rho, \varphi, z\}$  (see Fig. on the left) may be defined by their relations with the Cartesian coordinates:



$$\begin{aligned} r_1 &= \rho \cos \varphi, \\ r_2 &= \rho \sin \varphi, \\ r_3 &= z. \end{aligned} \quad (10.1)$$

- Gradient of a scalar function:

$$\nabla f = \mathbf{n}_\rho \frac{\partial f}{\partial \rho} + \mathbf{n}_\varphi \frac{1}{\rho} \frac{\partial f}{\partial \varphi} + \mathbf{n}_z \frac{\partial f}{\partial z}. \quad (10.2)$$

- The Laplace operator of a scalar function:

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}, \quad (10.3)$$

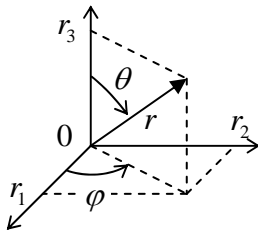
- Divergence of a vector function:

$$\nabla \cdot \mathbf{f} = \frac{1}{\rho} \frac{\partial(\rho f_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial f_\varphi}{\partial \varphi} + \frac{\partial f_z}{\partial z}. \quad (10.4)$$

- Curl of a vector function:

$$\nabla \times \mathbf{f} = \mathbf{n}_\rho \left( \frac{1}{\rho} \frac{\partial f_z}{\partial \varphi} - \frac{\partial f_\varphi}{\partial z} \right) \frac{\partial f}{\partial \rho} + \mathbf{n}_\varphi \left( \frac{\partial f_\rho}{\partial z} - \frac{\partial f_z}{\partial \rho} \right) + \mathbf{n}_z \left( \frac{\partial(\rho f_\varphi)}{\partial \rho} - \frac{\partial f_\rho}{\partial \varphi} \right). \quad (10.5)$$

(ii) Spherical coordinates  $\{r, \theta, \varphi\}$  (see Fig. on the left) may be defined as:



$$\begin{aligned} r_1 &= r \sin \theta \cos \varphi, \\ r_2 &= r \sin \theta \sin \varphi, \\ r_3 &= r \cos \theta. \end{aligned} \quad (10.6)$$

- Gradient of a scalar function:

<sup>10</sup> Some other orthogonal coordinate systems are discussed in EM Sec. 2.3.

<sup>11</sup> In 2D geometry with fixed coordinate  $z$ , these coordinates are called “polar”.

$$\nabla \mathbf{f} = \mathbf{n}_r \frac{\partial f}{\partial r} + \mathbf{n}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \mathbf{n}_\varphi \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi}. \quad (10.7)$$

- The Laplace operator of a scalar function:<sup>10</sup>

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{(r \sin \theta)^2} \frac{\partial^2 f}{\partial \varphi^2}. \quad (10.8)$$

- Divergence of a vector function:

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial (r^2 f_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (f_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial f_\varphi}{\partial \varphi}. \quad (10.9)$$

- Curl of a vector function:

$$\nabla \times \mathbf{f} = \mathbf{n}_r \frac{1}{r \sin \theta} \left( \frac{\partial (f_\varphi \sin \theta)}{\partial \theta} - \frac{\partial f_\theta}{\partial \varphi} \right) + \mathbf{n}_\theta \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial f_r}{\partial \varphi} - \frac{\partial (r f_\varphi)}{\partial r} \right) + \mathbf{n}_\varphi \frac{1}{r} \left( \frac{\partial (r f_\theta)}{\partial r} - \frac{\partial f_r}{\partial \theta} \right). \quad (10.10)$$

## 11. Products involving vector $\nabla$

(i) Useful zeros:

- For any scalar function  $f(\mathbf{r})$ ,

$$\nabla \times (\nabla f) \equiv \mathbf{curl}(\mathbf{grad} f) = 0. \quad (11.1)$$

- For any vector function  $\mathbf{f}(\mathbf{r})$ ,

$$\nabla \cdot (\nabla \times \mathbf{f}) \equiv \mathbf{div}(\mathbf{curl} \mathbf{f}) = 0. \quad (11.2)$$

(ii) Laplace operator via the curl of curl:

$$\nabla^2 \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla \times (\nabla \times \mathbf{f}). \quad (11.3)$$

(iii) Differentiation of a product of a scalar function by a vector function:

- The scalar 3D generalization of Eq. (4.1) is

$$\nabla \cdot (f \mathbf{g}) = (\nabla f) \cdot \mathbf{g} + f(\nabla \cdot \mathbf{g}), \quad (11.4a)$$

and its vector generalization is similar:

$$\nabla \times (f \mathbf{g}) = (\nabla f) \times \mathbf{g} + f(\nabla \times \mathbf{g}). \quad (11.4b)$$

(iv) 3D differentiation of products of two vector functions:

- The most important formula of this class is

$$\nabla \times (\mathbf{f} \times \mathbf{g}) = \mathbf{f}(\nabla \cdot \mathbf{g}) - (\mathbf{f} \cdot \nabla) \mathbf{g} - (\nabla \cdot \mathbf{f}) \mathbf{g} + (\mathbf{g} \cdot \nabla) \mathbf{f}, \quad (11.5)$$

but the following two formulas also may be met:

$$\nabla \cdot (\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \nabla) \mathbf{g} + (\mathbf{g} \cdot \nabla) \mathbf{f} + \mathbf{f} \times (\nabla \times \mathbf{g}) + \mathbf{g} \times (\nabla \times \mathbf{f}), \quad (11.6)$$

$$\nabla \cdot (\mathbf{f} \times \mathbf{g}) = (\nabla \times \mathbf{f}) \cdot \mathbf{g} - \mathbf{f} \cdot (\nabla \times \mathbf{g}). \quad (11.7)$$

## 12. Integro-differential relations

(i) For an arbitrary surface  $A$  limited by closed contour  $C$ :

- The Stokes theorem:

$$\int_A (\nabla \times \mathbf{f}) \cdot d\mathbf{A} \equiv \int_A (\nabla \times \mathbf{f})_n d^2 r = \oint_C \mathbf{g} \cdot d\mathbf{r} \equiv \oint_C g_\tau dr, \quad (12.1)$$

where  $d\mathbf{A} = \mathbf{n}d^2r$  is the elementary area vector (normal to the surface), and  $d\mathbf{r}$  is the elementary contour length vector (tangential to the contour line).

(ii) For an arbitrary volume  $V$  limited by closed surface  $A$ :

- Divergence theorem:

$$\int_V (\nabla \cdot \mathbf{f}) d^3 r = \oint_A \mathbf{f} \cdot d\mathbf{A} \equiv \oint_A f_n d^2 r. \quad (12.2)$$

- Green's theorem for two arbitrary scalar functions  $f$  and  $g$ :<sup>12</sup>

$$\int_V (f \nabla^2 g - g \nabla^2 f) d^3 r = \oint_A (f \nabla g - g \nabla f) \cdot d^2 r. \quad (12.3)$$

## 13. The Kronecker delta and Levi-Civita symbols

- The Kronecker delta-symbol (defined for integer indices):

$$\delta_{jj'} \equiv \begin{cases} 1, & \text{if } j' = j, \\ 0, & \text{otherwise.} \end{cases} \quad (13.1)$$

- The Levi-Civita permutation symbol (most frequently used for 3 integer indices, each taking values 1, 2, or 3):

$$\varepsilon_{jj'j''} \equiv \begin{cases} +1, & \text{if all 3 indices are different and follow in any of "correct" orders : 123, 231, or 312,} \\ -1, & \text{if all 3 indices are different and follow in any of "incorrect" orders : 321, 213, or 132,} \\ 0, & \text{if any pair of indices coincide.} \end{cases} \quad (13.2)$$

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<sup>12</sup> This formula readily follows from the divergence theorem (12.2) by applying it to vector functions  $f\nabla g$  and  $g\nabla f$ , then using Eq. (11.4a) in the left-hand parts of each result, and subtracting them.

### 14. Dirac's delta function

- Definition of 1D  $\delta$ -function (for real  $a < b$ ):

$$\int_a^b f(x)\delta(x)dx = \begin{cases} f(0), & \text{if } a < 0 < b, \\ 0, & \text{otherwise,} \end{cases} \quad (14.1)$$

where  $f(x)$  is any function continuous near  $x = 0$ . In particular (if  $f(x) = 1$  near  $x = 0$ ), the definition yields

$$\int_a^b \delta(x)dx = \begin{cases} 1, & \text{if } a < 0 < b, \\ 0, & \text{otherwise.} \end{cases} \quad (14.2)$$

- A very important integral:

$$\int_{-\infty}^{+\infty} e^{i\omega t} dt = 2\pi\delta(\omega), \quad (14.3a)$$

where  $i$  is the imaginary unity. The coefficient in this equation may be readily checked (or recalled:-) by considering it the Fourier-integral presentation of  $f(t) \equiv 1$ , and applying Eq. (14.1) to the reciprocal Fourier transform

$$1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} [2\pi\delta(\omega)] d\omega. \quad (14.3b)$$

- 3D generalization of the delta-function (the 2D generalization is similar):

$$\int_V f(\mathbf{r})\delta(\mathbf{r})d^3r = \begin{cases} f(0), & \text{if } 0 \in V, \\ 0, & \text{otherwise;} \end{cases} \quad (14.4)$$

its presentation via a product of 1D delta-functions of Cartesian coordinates:

$$\delta(\mathbf{r}) = \delta(r_1)\delta(r_2)\delta(r_3). \quad (14.5)$$

### 15. The Cauchy integral

- For any complex function  $f(w)$  which is analytical within the part of the complex plane  $w$ , which is limited by closed contour  $C$  and includes point  $w = z$ ,

$$f(z) = \frac{1}{2\pi i} \oint_C f(w) \frac{dw}{w-z}. \quad (15.1)$$

### 16. References

(i) For more formulas, and their discussions, I can recommend the following handbooks (in the alphabetic order):

- M. Abramowitz and I. S. Stegun, *Handbook of Mathematical Formulas*, Dover, 1965,

- I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 5<sup>th</sup> ed., Academic Press, 1980,
- A. Jeffrey and H. H. Dai, *Handbook of Mathematical Formulas and Integrals*, 4<sup>th</sup> ed., Academic Press, 2008,
- G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, 2<sup>nd</sup> ed., Dover, 2000.

Many formulas are also available from the symbolic calculation parts of commercially available software packages listed below.

On a personal note, perhaps 90% of all formula needs during my 45+ year research career have been satisfied by a small, wonderfully compiled book:

- H. B. Dwight, *Tables of Integrals and Other Mathematical Formulas*, 4<sup>th</sup> ed., MacMillan, 1961, whose used copies, rather surprisingly, are still available on the Web.

(ii) Properties of some special functions are briefly discussed in the most relevant points of the lecture notes:

- Bessel functions: EM Sec. 2.4;
- Hermite polynomials: QM Sec. 2.6;
- Laguerre polynomials (both simple and associated): QM Sec. 3.5;
- Legendre polynomials, associated Legendre functions, and spherical harmonics: EM Sec. 2.4 and QM Sec. 3.5.

(iii) Perhaps the most popular code collections for numerical calculations are the twin manuals

- W. H. Press *et al.*, *Numerical Recipes in FORTRAN*, 2<sup>nd</sup> ed., Cambridge U. Press, 1992;
- W. H. Press *et al.*, *Numerical Recipes [in C++ - KKL]*, 3<sup>rd</sup> ed., Cambridge U. Press, 2007.

My lecture notes include only very brief introductions into numerical methods of solutions of differential equations:

- ordinary differential equations: CM Sec. 3.9;
- equations with partial derivatives: CM Sec. 3.4 and EM Sec. 2.7.

(iv) The most popular software packages for numerical and symbolic calculations (in the alphabetic order):

- *Maple* (the current official Web site: <http://www.maplesoft.com/>);
- *Mathcad* (<http://www.ptc.com/products/mathcad/>);
- *Mathematica* (<http://www.wolfram.com/products/mathematica/index.html>);
- *MATLAB* (<http://www.mathworks.com/products/matlab/>).