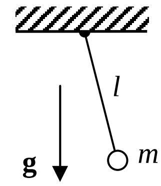
PHY 501 **Classical Mechanics and Dynamics** Fall 2009

Midterm exam problems with solutions



Problem M.1 (200 points). For a stretchable pendulum (i.e. a mass on a spring which exerts force $F = -k(l - l_0)$, where k and l_0 are positive constants), whose motion is confined to a vertical plane passing through the suspension point:

(i) calculate the Lagrangian function;

(ii) write down the Lagrangian equations of motion,

(iii) calculate the Hamiltonian function H; find out whether it is conserved,

(iv) calculate energy E; is $E = H$?; is energy conserved?

Solution: The kinetic and gravity energies here are similar to those of the top pendulum in the previous problem, so that

$$L = K - U = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) + mgl\cos\theta - \frac{k}{2}(l - l_0)^2 + \text{const.}$$

From here, the Lagrangian equations of motion are:

$$\ddot{l} + \omega_0^2(l - l_0) - l\dot{\theta}^2 - g\cos\theta = 0,$$
$$l\ddot{\theta} + 2\dot{l}\dot{\theta} + g\sin\theta = 0, \quad \omega_0^2 \equiv k/m.$$

Since K is a quadratic-homogeneous function of $\dot{l}$ and $\dot{\theta}$, H equals the total energy E:

$$H = E = K + U = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) - mgl\cos\theta + \frac{k}{2}(l - l_0)^2 + \text{const},$$

and since $\partial L/\partial t = 0,$ both are conserved.

Problem M.2 (150 points). A particle is launched, from afar, with impact parameter b, toward an attracting center with central potential

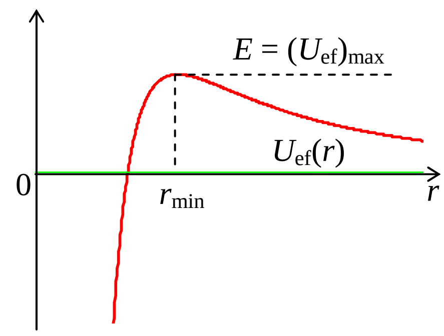
$$U(r) = -\frac{\alpha}{r^n}, \quad \text{with } n > 2, \alpha > 0.$$

The initial kinetic energy E of the particle is barely sufficient for escaping the capture by the attracting center. Express the minimum distance between the particle and the center via b.

Solution: The above condition on the initial energy means that E should be equal to the maximum of the effective energy of the radial motion,

$$U_{\text{ef}}(r) \equiv U(r) + \frac{L_z^2}{2mr^2} = -\frac{\alpha}{r^n} + \frac{L_z^2}{2mr^2},$$

(see Fig. on the right). With account of Eq. (3.66) of the lecture notes, this formula turns into



$$U_{\text{ef}}(r) = -\frac{\alpha}{r^n} + \frac{Eb^2}{r^2}. \qquad (*)$$

It is evident that $r_{\min}$ may be found from equation $dU_{\text{ef}}/dr = 0$. Differentiating Eq. (*), we find that

$$r_{\min} = \left(\frac{n\alpha}{2Eb^2}\right)^{\frac{1}{n-2}}, \qquad (**)$$

$$(U_{\text{ef}})_{\max} \equiv U_{\text{ef}}(r_{\min}) = \left(\frac{2Eb^2}{n\alpha}\right)^{\frac{2}{n-2}} Eb\frac{n-2}{n}.$$

Now, condition $E = (U_{\text{ef}})_{\max}$ yields

$$\left(\frac{2Eb^2}{n\alpha}\right)^{\frac{1}{n-2}} = \frac{1}{b}\left(\frac{n}{n-2}\right)^{\frac{1}{2}}.$$

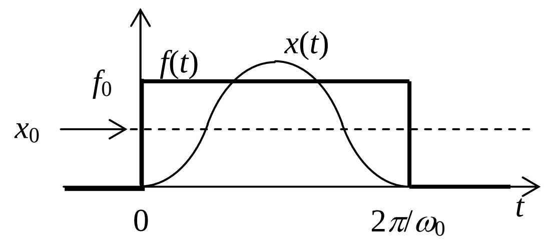
Plugging this result into Eq. (**), we finally get

$$r_{\min} = b\left(\frac{n-2}{n}\right)^{1/2} < b.$$

Problem M.3 (200 points). A square-wave pulse of force (see the Fig. on the right) is exerted on a harmonic oscillator with eigenfrequency ω_0 (no damping), initially at rest. Calculate the law of motion $x(t)$, sketch it, and interpret the result.

Solution: Following the Green's function method, we may express the solution as

$$x(t) = \int_0^\infty f(t-\tau)G(\tau)d\tau. \qquad (*)$$



The Green's function of an oscillator has been calculated in class – see Eq. (4.34) of the lecture notes. In the limit of negligible damping δ,

$$G(\tau) = \frac{\sin\omega_0\tau}{\omega_0}.$$

Due to the piecewise-constant character of function $f(t)$, the non-vanishing parts of integral (*) have different limits (and hence give different final results) for the cases $0 < t < 2\pi/\omega_0$:

$$x(t) = f_0\int_0^t G(\tau)d\tau = x_0(1-\cos\omega_0 t),$$

and $t > 2\pi/\omega_0$:

$$x(t) = f_0 \int_{t-2\pi/\omega_0}^{t} G(\tau)d\tau = 0.$$

The final result is sketched with a thin line in Fig. above. Its physics is simple: the step of force applied at $t = 0$ shifts the equilibrium position from 0 to $x_0 = f_0/\omega_0^2 = F_0/k$. Hence at $t = 0$ the oscillator is out of this new equilibrium point, and starts periodic oscillations around the new equilibrium position with amplitude $A = x_0$. The equal and opposite step of force, arriving at time $t = 2\pi/\omega_0$, quenches these oscillations completely. (If the time interval between the two steps was not exactly a multiple of the oscillation period, the compensation would not be complete.)

Problem M.4 (150 points). Find the fixed point of the following system of equations:

$$\dot{x} = -x - 4y,$$
$$\dot{y} = -y - x.$$

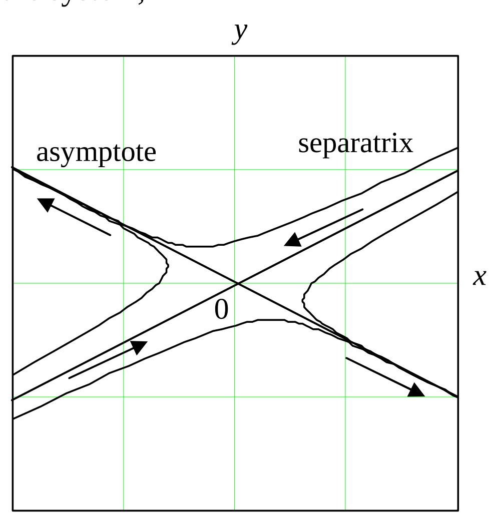
Analyze the point's stability. What type of the fixed point is it? (node? saddle? focus? center?) Sketch the phase plane $[x,y]$ in as much detail as you can.

Solution: This system of equations is linear, so it has only one (trivial) fixed point $x = y = 0$, and no linearization is necessary. Solving the characteristic equation of the system,

$$\begin{vmatrix} -1-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = 0,$$

we see that both characteristic exponents are real: $\lambda_\pm = -1 \pm 2$. One of them (λ_+) is positive and another negative, so that the fixed point is the (unstable) saddle.

Figure on the right shows several trajectories on the phase plane of this system. The slopes of the separatrix and asymptote may be found by plugging the values of, respectively, λ_- and λ_+ back into the initial system of equations. This yields



$$y/x\Big|_{\text{separatrix}} = 1/2, \quad y/x\Big|_{\text{asymptote}} = -1/2,$$

in agreement with the plots.