PHY 506 **Classical Electrodynamics II** Spring 2011

Final Exam with solutions

Problem F.1 (to be graded of 200 points). Use the Born approximation to calculate the differential cross-section of scattering of a plane wave normally incident on a very thin, round dielectric disk of radius R. Analyze the result in the limits $kR << 1$ and $kR >> 1$.

Solution: This problem is just a particular case of Optional Problem O.10, with $q_z L << 1$, and $q_\perp = k\sin\alpha$, where α is the scattering angle between vectors $\mathbf{k}$ and $\mathbf{k}_0$. (Note that it is different from angle θ between vectors $\mathbf{k}$ and $\mathbf{E}$!) Thus we can immediately write the differential cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{k^4 L^2 R^4}{16}(\varepsilon_r - 1)^2 \sin^2\theta \left(\frac{J_1(kR\sin\alpha)}{(kR\sin\alpha)/2}\right)^2 . \qquad (*)$$

If the disk radius is small ($kR << 1$), this result is reduced to Eq. (8.53), with $V = \pi R^2 d$. Taking into account that

$$\oint_{4\pi} \sin^2\theta d\Omega = 2\pi \int_0^\pi \sin^3\theta d\theta = \frac{8\pi}{3},$$

the full cross-section, in this limit, is

$$\sigma = \frac{\pi}{6} k^4 L^2 R^4 (\varepsilon_r - 1)^2 .$$

For a large disk ($kR >> 1$), the last factor in Eq. (*) limits scattering to small forward and back angles, with $|\sin\alpha| \sim 1/kR << 1$, and $\sin\theta \approx 1$, so that σ may be calculated as

$$\sigma = 2\int_{\alpha<<1} \frac{d\sigma}{d\Omega} d\Omega = 4\pi \int_{\alpha<<1} \frac{d\sigma}{d\Omega}\alpha d\alpha = 4\pi \frac{k^4 L^2 R^4}{16}(\varepsilon_r - 1)^2 \int_0^\infty \left(\frac{J_1(kR\alpha)}{(kR\alpha)/2}\right)^2 \alpha d\alpha$$

$$= 4\pi \frac{k^4 L^2 R^4}{16}(\varepsilon_r - 1)^2 4\int_0^\infty \frac{J_1^2(\xi)}{\xi} d\xi = \frac{\pi}{2} k^4 L^2 R^4 (\varepsilon_r - 1)^2 .$$

Thus, σ is described by essentially the same formula as at $kR << 1$, just with a different numerical coefficient.

Problem F.2 (300 points). A relativistic particle with energy $\mathcal{E}$ and rest mass m collides with a similar particle, initially at rest in the laboratory reference frame. Find:

(i) the velocity of the center of mass of the system, in the lab frame,
(ii) the total energy of the system, in the center-of-mass frame, and
(iii) final velocities of both particles (in the lab frame), if they move along the same direction.

Solutions:

(i) Before the collision, the total momentum p_s of the system in the lab frame equals that (p) of the only moving particle:

$$p_s = p = \left[(\mathcal{E}/c)^2 - (mc)^2\right]^{1/2},$$

while the total energy is

$$\mathcal{E}_\mathrm{s} = \mathcal{E} + mc^2 .$$

By definition, the center of mass of a system is an imaginary particle having the same total momentum $\boldsymbol{p}_s$ and total mass M_s as the system. (In relativity, the latter is the dynamic mass $M_s = \mathcal{E}_s/c^2$.) Applying to the c.o.m. the general formula discussed in class, $\boldsymbol{p} = M\mathbf{u}$, i.e. $\boldsymbol{\beta} = \boldsymbol{p}/Mc = \boldsymbol{p}c/\mathcal{E}$, we get[1]

$$\beta_\mathrm{s} = \frac{\left[(\mathcal{E}/c)^2 - (mc)^2\right]^{1/2} c}{\mathcal{E} + mc^2} = \left(\frac{\mathcal{E} - mc^2}{\mathcal{E} + mc^2}\right)^{1/2}, \qquad \text{so that } \gamma_\mathrm{s} = \frac{1}{\left(1 - \beta_s^2\right)^{1/2}} = \left(\frac{\mathcal{E} + mc^2}{2mc^2}\right)^{1/2}. \qquad (*)$$

Notice that despite the particles are similar, $\beta_\mathrm{s} = \beta/2$ only in the non-relativistic limit; in the opposite, ultra-relativistic limit ($\beta \approx 1$, $\mathcal{E} >> mc^2$), the c.o.m. velocity is also close to c.

(ii) Applying the Lorentz transform, valid for an arbitrary 4-vector, to the 0th component of the total 4-momentum of the system, $p_\mathrm{s}^\alpha = \{\mathcal{E}_\mathrm{s}/c, p_\mathrm{s}, 0, 0\}$, we have

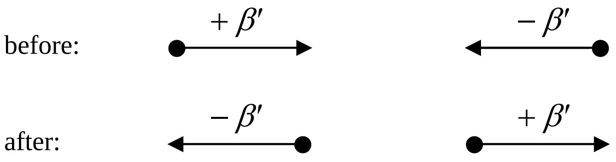
$$\frac{(\mathcal{E}_s)_\mathrm{com}}{c} = \gamma_s\left(\frac{\mathcal{E}_\mathrm{s}}{c} - \beta_\mathrm{s} p_\mathrm{s}\right) = \gamma_s\left(\frac{\mathcal{E} + mc^2}{c} - \beta_\mathrm{s}\left[\left(\frac{\mathcal{E}}{c}\right)^2 - (mc)^2\right]^{1/2}\right).$$

Plugging in β_s and γ_s from Eq. (*), we finally get[2]

$$(\mathcal{E}_s)_\mathrm{com} = 2\gamma_\mathrm{s} mc^2 = \left[2mc^2(\mathcal{E} + mc^2)\right]^{1/2} . \qquad (**)$$

Notice that for ultra-relativistic particles, $\mathcal{E} \approx mc^2$, the c.o.m. energy (which is the one crucial for new particle generation) grows only as the square root of that in the lab frame; this is why particle colliders are more popular nowadays than the fixed-target accelerators. (The caveat is that colliders should have very high beam density to ensure a sufficient collision rate.)

(iii) By the definition of the c.o.m. frame, velocities of 2 similar particles, observed from it, should be equal and opposite, and since the energy is conserved at collision, their magnitudes β' should be the same both before and after it. At the direct collision, with particles moving along the same direction as before it, this means that particles just exchange their velocities - see Fig. below.



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[1] Alternatively, this result can be obtained via the Lorentz transform of the 1st component of the 4-momentum of the system, just as it is done for its 0th component in part (ii) below.

[2] Another (perhaps, even simpler) way to get the same result is to require the norm of the total 4-momentum of the system, $p^\alpha p_\alpha$, to be Lorentz invariant:

$$\frac{(\mathcal{E}_s)^2_\mathrm{com}}{c^2} - (p_s)^2_\mathrm{com} = \frac{(\mathcal{E}_\mathrm{s})^2}{c^2} - p_s^2 ,$$

and notice that the net momentum in the c.o.m. frame, $(p_s)_\mathrm{com}$, is zero.

But this fact (of particle exchanging velocities) should be true in *any* reference frame. Hence, in the lab system the final velocities are also the same as the initial ones:

$$\beta_+ = \beta = \frac{cp}{\mathcal{E}} = \frac{1}{\mathcal{E}}\left[\mathcal{E}^2 - (mc^2)^2\right]^{1/2}, \qquad \beta_- = 0.$$

Problem F.3 (400 points). Find the law of motion of a relativistic particle in parallel (uniform and static) electric and magnetic fields.

Hint: You may like to use the proper time of the particle.

Solution: For an arbitrary direction of the initial momentum $\boldsymbol{p}(0)$ of the particle, we can always select a coordinate frame so that

$$\mathbf{E} = \{0, 0, E\}, \quad \mathbf{B} = \{0, 0, B\}, \quad \boldsymbol{p}(0) = \{p_x(0), 0, p_z(0)\}.$$

Then the 4-vector equation (9.145) of motion of the particle,

$$\frac{du^\alpha}{d\tau} = \frac{q}{m} F^{\alpha\beta} u_\beta ,$$

may be separated into components as

$$\frac{d(\gamma c)}{d\tau} = \Gamma(\gamma u_z), \quad \frac{d(\gamma u_x)}{d\tau} = -\omega(\gamma u_y), \quad \frac{d(\gamma u_y)}{d\tau} = \omega(\gamma_u u_x), \quad \frac{d(\gamma u_z)}{d\tau} = \Gamma(\gamma c), \qquad (*)$$

where $\Gamma \equiv qE/cm$, and $\omega \equiv qB/m$. (Note that the last expression is *not* the usual relativistic cyclotron frequency (9. 151), $\omega_c = qB/\gamma m$: its denominator does not include the Lorentz factor $\gamma = (1 - \beta^2)^{1/2}$, as it would if we dealt with the lab time t).

Now we see that the first and the last equations of Eqs. (*) form an independent system which may be readily reduced to one equation (by the usual trick of differentiation of one of them over τ, and then plugging the counterpart equation into the result).[3] As a result we get two similar equations,

$$\frac{d^2}{d\tau^2}(\gamma c) = \Gamma^2(\gamma c), \quad \frac{d^2}{d\tau^2}(\gamma u_z) = \Gamma^2(\gamma u_z),$$

which may be readily integrated, and integration constants found from the initial values of γ and u_z. The result is

$$\gamma(\tau) = \gamma(0)\left[\cosh\Gamma\tau + \beta_z(0)\sinh\Gamma\tau\right], \quad u_z(\tau) = c\,\frac{\beta_z(0)\cosh\Gamma\tau + \sinh\Gamma\tau}{\cosh\Gamma\tau + \beta_z(0)\sinh\Gamma\tau}. \qquad (**)$$

Now the second and third of Eqs. (*) form a similar system (in which the electric field does not participate), which is exactly the same as for the nonrelativistic cyclotron motion. It may be readily integrated to give (with the appropriate choice of the origin of proper time τ, and axis y in plane of the initial velocity):

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[3] Note that since this system does not depend on magnetic field, this is just an alternative way of solving the problem discussed in Sec. 9.6(ii) and Homework Problem 7.2.

$$\gamma u_x = -\frac{p_x(0)}{m}\sin\omega\tau, \quad \gamma u_y = \frac{p_x(0)}{m}\cos\omega\tau .$$

Since $\gamma u_x = \gamma dx/dt = dx/d\tau$ (and similarly for y), these equations may be readily integrated again, giving

$$x = R\cos\omega\tau + \text{const}, \quad y = R\cos\omega\tau + \text{const} ,$$

where the cyclotron orbit radius $R \equiv p_y(0)/qB$ is not affected by the initial momentum of the particle in z direction.

The relation between proper time τ and lab time t may be found by plugging the first of Eqs. (**) into the fundamental relation $dt = \gamma d\tau$, and integrating it. The result,

$$t = \frac{\gamma(0)}{\Gamma}\left[\sinh\Gamma\tau + \beta_z(0)\cosh\Gamma\tau\right] + \text{const} ,$$

shows that at large times ($\tau >> 1/\Gamma$), proper time is a logarithm of the lab time, so that as viewed from the laboratory frame, the cyclotron motion gradually slows down, due to the gradual increase of the effective mass $M = \gamma m$ of the particle because of the increase of its energy by the electric field.

Problem F.4 (200 points). A TEM wave propagates along a coaxial cable. Neglecting the skin depth in comparison with the cross-section dimensions, find the pressure imposed by the EM field of the wave on conductor surfaces. Interpret the result. Qualitatively, how would it be affected by small Ohmic losses in the conductors?

Solution: In TEM waves propagating without attenuation, the electric and magnetic field are related by the same Eq. (7.6),

$$H = \frac{E}{Z} = \left(\frac{\varepsilon}{\mu}\right)^{1/2} E, \quad \text{i.e.}\, B = \mu H = (\varepsilon\mu)^{1/2} E ,$$

as in the plane waves, at all points of the transmission line's cross-section, in particular on conductors' surfaces. As the result, the positive pressure (9.242) of the magnetic field and the negative pressure (9.240), imposed by the electric field, compensate each other:

$$\frac{dF}{dA} = \frac{B^2}{2\mu} - \frac{\varepsilon E^2}{2} = \frac{\varepsilon\mu E^2}{2\mu} - \frac{\varepsilon E^2}{2} = 0 .$$

This result may be readily interpreted as the corollary of the Poynting vector **S** having no transversal components. Because of that, the EM field momentum $\mathbf{g} = \mathbf{S}/c^2$ is directed along the cable, i.e. tangential to the surfaces of its conductors, and does not transfer to them any momentum.

At small but nonvanishing power dissipation, **S** (and hence **g**) gradually decrease along cable's length. This momentum change may be only due to interaction with conductor surfaces, so that one can expect the wave to impose a tangential (rather than normal) force on the walls, in the direction of wave propagation.

Problem F.5 (300 points). Find polarization of the synchrotron radiation propagating

(i) within the particle rotation plane, and

(ii) perpendicular to the plane.

Solution: The second term of the general Eq. (10.20a) of the lecture notes shows that the instant electric field **E** of the radiated wave is oriented as vector

$$\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\frac{d\boldsymbol{\beta}}{dt}\right]$$

at the retarded point, i.e. at the charge position. First of all, this relation shows that (as in each locally-plane wave in free space), that vector **E** is perpendicular to wave's propagation direction **n**, so that we "only" need to find the direction of **E** in the plane perpendicular to **n**.

At synchrotron radiation, vectors $\boldsymbol{\beta}$ and $d\boldsymbol{\beta}/dt$ are always in the plane of particle's rotation, and perpendicular to each other. This is why in the observer (vector **n**) is also in that plane, the inner product vector $(\mathbf{n} - \boldsymbol{\beta})\times d\boldsymbol{\beta}/dt$ is perpendicular to that plane, so that the outer product vector, and hence vector **E** are in the rotation plane again. (This conclusion could be also made from the evident mirror symmetry of the problem relative to the rotation plane.)

In the opposite limit, for radiation propagating perpendicular to the rotation plane, it is beneficial to present the double product as the sum of two terms:

$$\mathbf{n}\times\left(\mathbf{n}\times\frac{d\boldsymbol{\beta}}{dt}\right)-\mathbf{n}\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{dt}\right),$$

because the second term vanishes. (Indeed, both the inner product vector $\boldsymbol{\beta}\times d\boldsymbol{\beta}/dt$ and **n** are perpendicular to the rotation plane, i.e. are parallel to each other.) The first term is opposite in direction to the instant $d\boldsymbol{\beta}/dt$, i.e. vector **E** rotates, within the plane, with the angular velocity ω_c of particle's motion. According to Eq. (10.61) of the lecture notes, radiation propagating in this direction has only this frequency component - even in the ultrarelativistic case when in-plane radiation has $\sim \gamma^3 >> 1$ harmonics! This is why that **E** rotates (about wave's direction **n**) with the monochromatic wave's frequency. This is exactly the circularly-polarized wave which was discussed in Sec. 7.1 – see Fig. 7.3b..