PHY 505 **Classical Electrodynamics I** Fall 2010

Final exam with solutions

Problem F.1 (to be graded of 200 points). In two separate experiments, a thin, plane sheet of a linear dielectric with ε = const is placed into a uniform external electric field $\mathbf{E}_0$:

(i) with the sheet surface parallel to the electric field, and

(ii) the surface perpendicular to the field.

For each case, find the electric field **E**, electric displacement **D**, and polarization **P** inside the dielectric.

Solution: The same reasoning which was used in class (see Sec. 3.4 of the lecture notes) to discuss vacuum slits in dielectrics, gives the following results:

(i) the sheet surface parallel to the electric field:

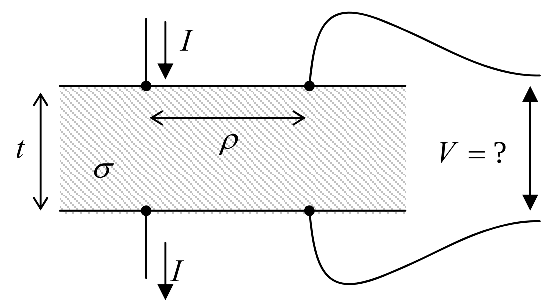
$$E = E_0, \quad D = \varepsilon E, \quad P = D - \varepsilon_0 E = (\varepsilon - \varepsilon_0)E_0 .$$

(ii) the surface perpendicular to the field:

$$D = D_0 = \varepsilon_0 E_0, \quad E = \frac{D}{\varepsilon} = \frac{\varepsilon_0 E_0}{\varepsilon} = \frac{E_0}{\varepsilon_r}, \quad P = D - \varepsilon_0 E = \frac{\varepsilon - \varepsilon_0}{\varepsilon}\varepsilon_0 E_0 = \frac{\varepsilon - \varepsilon_0}{\varepsilon_r} E_0 .$$

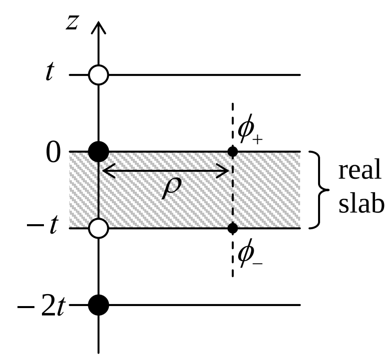
We see that in the latter case all fields are ε_r times weaker than in the first one.

Problem F.2 (400 points). Calculate voltage drop V across a uniform, broad, resistive slab of thickness t, at distance ρ from the points of injection/ejection of dc current I which is passed across the slab (see Fig. on the right).



Hint: A (correct :-) answer in the form of a series is acceptable.

Solution: A has been discussed in class (see Sec. 4.3 of the lecture notes), both the Laplace equation for the electrostatic potential ϕ inside the conductor (related to the dc current density as $\mathbf{j} = \sigma\mathbf{E} = -\sigma\nabla\phi$), and boundary conditions outside the injection/ejection points ($j_n = -\sigma\partial\phi/\partial n = 0$) at its surface may be satisfied by the complementing the real slab by an infinite stack of slabs, filling the whole space, with a periodic pattern of mirror images of the injection/ejection points – see Fig. on the right. With the coordinates selected as shown in this figure, points of current injection are located at $z_k = 2kt$, while points of current ejection at $z_k' = (2k - 1)t$, where k is an integer). Since the magnitude of current in each point is $2I$ (as it is evident, for example, from Fig. 4.7 with $d \to 0$), it creates, in the slab stack, a spherically-symmetric distribution of current density and electric field:



$$\mathbf{j}_k(\mathbf{r}) = \frac{2(-1)^k I}{4\pi|\mathbf{r}_k - \mathbf{r}|^3}(\mathbf{r}_k - \mathbf{r}), \qquad \mathbf{E}_k(\mathbf{r}) = \frac{\mathbf{j}_k}{\sigma} = \frac{(-1)^k I}{2\pi\sigma|\mathbf{r}_k - \mathbf{r}|^3}(\mathbf{r}_k - \mathbf{r}),$$

and hence a spherically symmetric potential distribution

$$\phi_k(z) = (-1)^k \frac{I}{2\pi\sigma|\mathbf{r}_k - \mathbf{r}|} = (-1)^k \frac{I}{2\pi\sigma\left[(kt-z)^2 + \rho^2\right]^{1/2}}.$$

Summing up contributions for all k, for the top point of voltage pick-up ($z = 0$) we have

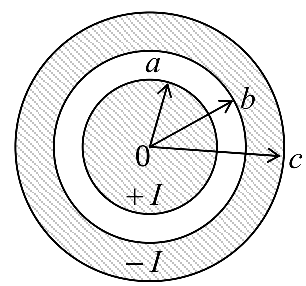
$$\phi_+ = \frac{I}{2\pi\sigma}\sum_{k=-\infty}^{+\infty}\frac{(-1)^k}{\left(k^2 + \rho^2\right)^{1/2}}.$$

From Fig. above, it is evident that the potential at the lower pick-up point is opposite, $\phi_- = -\phi_+$, so that the voltage drop V is

$$V = \phi_+ - \phi_- = 2\phi_+ = \frac{I}{\pi\sigma}\sum_{k=-\infty}^{+\infty}\frac{(-1)^k}{\left(k^2t^2 + \rho^2\right)^{1/2}}.$$

At $\rho/t \to 0$, the voltage diverges as $1/\rho$, reflecting the potential divergence near the injection/ejection points. (This divergence would naturally disappear at an account of a finite size of current electrodes.) In the opposite limit $\rho/t >> 1$, voltage decreases with distance as $\exp\{-\pi\rho/t\}$.

Problem F.3 (300 points). Find self-inductance (per unit length) of a long coaxial cable with the cross-section shown in the Fig. on the right, provided that current I is uniformly distributed over the cross-sections of both conductors.



Solution: The most straightforward way to calculate the inductance is again from the magnetic energy, using Eq. (5.79) of the lecture notes. For that, we need first to calculate the radial distribution of the magnetic field (which of course has only one, azimuthal component, because of the axial symmetry of the problem). This distribution may be immediately found from the application of the Ampère law to circles of radii ρ within four different ranges:

$$2\pi\rho B = \mu_0 I\Big|_{\text{piercing the circle area}} = \mu_0 I \times \begin{cases} \dfrac{\rho^2}{a^2}, & \text{for } \rho < a, \\ 1, & \text{for } a < \rho < b, \\ \dfrac{c^2 - \rho^2}{c^2 - b^2}, & \text{for } b < \rho < c, \\ 0, & \text{for } c < \rho. \end{cases}$$

Now, an elementary integration yields:

$$\frac{U}{l}=\frac{1}{2\mu_0}\int B^2 d^2r=\frac{\pi}{\mu_0}\int_0^\infty B^2\rho d\rho=\frac{\mu_0 I^2}{4\pi}\left[\int_0^a\left(\frac{\rho}{a^2}\right)^2\rho d\rho+\int_a^b\left(\frac{1}{\rho}\right)^2\rho d\rho+\int_b^c\left(\frac{c^2-\rho^2}{\rho(c^2-b^2)}\right)^2\rho d\rho\right]$$

$$=\frac{\mu_0 I^2}{4\pi}\left[\frac{1}{4}+\ln\frac{b}{a}+\frac{1}{\left(c^2-b^2\right)^2}\left\{c^4\ln\frac{c}{b}-c^2\left(c^2-b^2\right)+\frac{c^4-b^4}{4}\right\}\right]$$

$$=\frac{\mu_0}{2\pi}\left[\ln\frac{b}{a}+\frac{c^2}{c^2-b^2}\left(\frac{c^2}{c^2-b^2}\ln\frac{c}{b}-\frac{1}{2}\right)\right]\frac{I^2}{2}.$$

From here, and Eq. (5.79), we get the final answer:

$$\frac{L}{l}=\frac{\mu_0}{2\pi}\left[\ln\frac{b}{a}+\frac{c^2}{c^2-b^2}\left(\frac{c^2}{c^2-b^2}\ln\frac{c}{b}-\frac{1}{2}\right)\right].$$

For the particular case of a thin outer conductor, $c - b \ll b$, this expression reduces to

$$\frac{L}{l}\approx\frac{\mu_0}{2\pi}\left(\ln\frac{b}{a}+\frac{1}{4}\right).$$

If, in addition, the inner conductor is also thin, $a \ll b$, we get a very simple result

$$\frac{L}{l}\approx\frac{\mu_0}{2\pi}\ln\frac{b}{a}$$

which describes the inductance due to the magnetic field energy in the free space between the conductors. This expression is very important for applications, because it is also valid (for *arbitrary* a, b, and c) at high frequencies, as well as for arbitrary frequencies in a superconductor cable with $a,(c-b) \gg \delta_L$.

Problem F.4 (350 points). Calculate the magnetic field distribution inside and outside a sphere made of a hard ferromagnet with permanent, uniform magnetization **M** = const.

*Solution***:** Due to the absence of macroscopic ("stand-alone") currents, magnetic field in the problem may be presented by Eq. (5.135) of the lecture notes, $\mathbf{H} = -\nabla\phi_m$, where the scalar potential ϕ_m satisfies the Laplace equation both inside the sphere and outside it. Due to the axial symmetry of the problem (about the direction of magnetization **M**, which we may take for the polar axis), the general solution of the Laplace equation may be presented in the form of Eq. (2.172) of the lecture notes. Setting the arbitrary constant a_0 to zero, and taking into account that due to zero divergence of vector **H**, constant b_0 is zero as well, we may reduce this solution to

$$\phi_m=\sum_{l=1}^{\infty}\mathcal{P}_l(\cos\theta)\times\begin{cases}\left(a_l r^l+b_l/r^{l+1}\right), & \text{for } r\le R\,,\\ \left(a_l' r^{l+1}+b_l'/r^{l+1}\right), & \text{for } r\le R\,.\end{cases}$$

where R is sphere's radius. Next, we may take into account that since the field is produced by the sphere itself, it should vanish at large distances from it, so that all coefficients a'_n must equal zero. Similarly, the potential inside the cylinder has to be finite, so that all coefficients b_n have to vanish as well. As a result, the above expression is reduced to

$$\phi_m = \sum_{l=1}^{\infty} \mathscr{P}_l(\cos\theta) \times \begin{cases} a_l r^l, & \text{for } r \le R, \\ b_l^{'} / r^{l+1}, & \text{for } r \le R, \end{cases}$$

where $\mathscr{P}_l(\xi)$ are the Legendre polynomials (2.169).

In order to find coefficients a_l and b'_l, we have to write boundary conditions at the sphere's surface ($r = R$). From Eq. (5.137) for the tangential component of the field, we get

$$\sum_{l=1}^{\infty} a_l R^l \mathscr{P}_l(\cos\theta) = \sum_{n=1}^{\infty} \frac{b_l^{'}}{R^{l+1}} \mathscr{P}_l(\cos\theta) . \qquad (*)$$

However, for the normal component of the field, we have to use the general Eq. (5.154), B_n = const (rather than Eq. (5.138) which is only valid for a linear magnetic), because **B** is a linear function of **H** (equal to μ_0**H**) only outside the sphere, at $r \ge R$. At $r \le R$ we should use the general relation (5.108), $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, for our geometry giving $B_n = \mu_0(H_n + M_n) = \mu_0(-\partial\phi_m/\partial r + M_n)$. For the fixed, uniform polarization, $M_n = M\cos\theta = M\mathscr{P}_1(\cos\theta)$, so that the second boundary condition takes the form

$$\mu_0\left[-\sum_{l=1}^{\infty} l a_l R^{l-1} \mathscr{P}_l(\cos\theta) + M\mathscr{P}_1(\cos\theta)\right] = \mu_0 \sum_{l=1}^{\infty} (l+1) \frac{b_l^{'}}{R^{l+2}} \mathscr{P}_l(\cos\theta) . \qquad (**)$$

Due to the mutual orthogonality of functions $\mathscr{P}_l(\cos\theta)$, Eqs. (*) and (**) fall apart into an infinite set of independent couples of linear equations for each pair $\{a_l, b'_l\}$, but only for $l = 1$ such a pair of equations is inhomogeneous, i.e. gives nonvanishing coefficients a_1, b'_1. (All other coefficients *may* equal zero, and due to the uniqueness of the Laplace equation solution, they *have to* equal zero.) Solving the system of equations for $l = 1$, we get

$$a_1 = \frac{M}{3}, \qquad b'_1 = \frac{MR^3}{3} .$$

Finally, the scalar potential of the magnetic field is

$$\phi_m\Big|_{\rho\le R} = \frac{M}{3} r\cos\theta, \quad \phi\Big|_{\rho\ge R} = \frac{MR^3}{3r^2}\cos\theta .$$

The first of these expressions describes a uniform magnetic field with $\mathbf{H} = -\mathbf{M}/3$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = (2/3)\mu_0\mathbf{M}$, while the second formula yields Eq. (5.99) for the field of a magnetic dipole with moment

$$\mathbf{m} = \frac{4\pi}{3} R^3 \mathbf{M}$$

– the result which could be guessed even without the formal solution of the problem.

Problem F.5 (300 points). Use the London equation to find the distribution of supercurrent density **j** across the circular cross-section (with radius $R \sim \delta_L$) of a long, straight superconducting wire which carries dc current I.

Solution: For this axially-symmetric problem, the solution of the London equation (see Eq. (6.44) of the lecture notes) may be looked for in the form $\mathbf{A}(\mathbf{r}) = A(\rho)\mathbf{n}_z$, where axis z coincides with that of the wire, and ρ is the distance of the observation point **r** from that axis. Using the well-known expression for the Laplace operator in cylindrical coordinates,[1] the equation becomes

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dA}{d\rho}\right)=\frac{1}{\delta_L^2}A,$$

where I have used definition (6.34) of the London's penetration depth. Introducing the dimensionless variable $\xi \equiv \rho/\delta_L$, and differentiating the left-hand part, this equation may be brought to the form,

$$\frac{d^2A}{d\xi^2}+\frac{1}{\xi}\frac{dA}{d\xi}-A=0,$$

which is the modified Bessel equation (2.155) with $\nu = 0$, with the general solution

$$A=a_1I_0(\xi)+a_2K_0(\xi),$$

so that the supercurrent density j, which is (within the London gauge) proportional to A, is

$$j(\rho)=c_1I_0\left(\frac{\rho}{\delta_L}\right)+c_2K_0\left(\frac{\rho}{\delta_L}\right).$$

Since the modified Bessel function of the second kind, $K_0(\xi)$ diverges at $\xi \to 0$ (see Eq. (2.157) and Fig. 2.20), while j has to stay finite at all points, coefficient c_2 has to equal 0. Coefficient c_1 may be found from the total current:

$$\int_A j(\rho)d^2\rho=2\pi\int_0^R j(\rho)\rho d\rho=2\pi c_1\int_0^R I_0\left(\frac{\rho}{\delta_L}\right)\rho d\rho=2\pi\,\delta_L^2c_1\int_0^{R/\delta_L}I_0(\xi)\xi d\xi=I.$$

The integral in the right-hand part may be calculated using recurrent relations (2.143) (valid for arbitrary Bessel functions) and their corollary $d[\xi^nI_n(\xi)]/d\xi = \xi^nI_{n-1}(\xi)$, in our current case with $n = 1$. As a result, we get

$$c_1=\frac{I}{2\pi\delta_L^2}\left[\int_0^{R/\delta_L}I_0(\xi)\xi d\xi\right]^{-1}=\frac{I}{2\pi\delta_L^2\left[\xi I_1(\xi)\right]_0^{R/\delta_L}}=\frac{I}{2\pi\delta_LRI_1(R/\delta_L)},\qquad j=\frac{I}{2\pi\delta_LR}\frac{I_0(\rho/\delta_L)}{I_1(R/\delta_L)}.$$

If the wire is very thin ($R \ll \delta_L$), then we may apply to this result the first or Taylor expansions (2.158) to write $I_0(\rho/\delta_L) \approx 1$, $I_1(R/\delta_L) \approx R/2\delta_L$, so that $j \approx I/\pi R^2$ = const, i.e. the supercurrent is uniformly distributed over the wire's cross-section. In the opposite limit $\delta_L \ll R$, we may use the first of asymptotic formulas (2.158) to reduce our result to

$$j\approx\frac{I}{2\pi\delta_LR}\left(\frac{R}{\rho}\right)^{1/2}\frac{\exp\{\rho/\delta_L\}}{\exp\{R/\delta_L\}}=\frac{I}{2\pi\delta_L(R/\rho)^{1/2}}\exp\left\{-\frac{R-\rho}{\delta_L}\right\}.$$

This expression shows that appreciable supercurrent only flows in a δ_L-thin sheet at the wire surface – exactly like in the plane-surface problem – see Eq. (6.45).

[1] See, e.g., MA Eq. (10.3).