

Problem 1.1 (to be graded of 25 points). A monochromatic, plane EM wave is normally incident from free space on a uniform slab of a material with electric permittivity ε and magnetic permeability μ , with slab thickness d comparable with the wavelength.

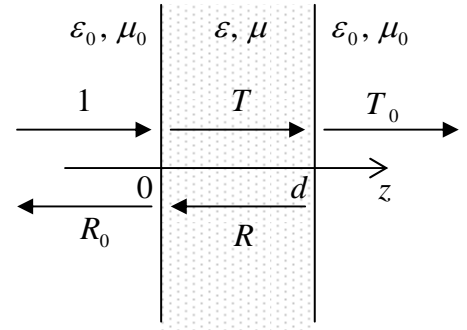
(i) Calculate the power transmission coefficient F , i.e. the fraction of the incident power, which is transmitted through the slab.

(ii) Assuming that ε and μ are frequency-independent and positive, analyze in detail the frequency dependence of F . In particular, how does function $F(\omega)$ depend on the film thickness d and the wave impedance $Z = (\mu/\varepsilon)^{1/2}$ of its material?

Solutions:

(i) Since the wave may be partly reflected from the second interface (see Fig. on the right), we should look for the solution in the form of five traveling waves:

$$E = E_\omega e^{-i\omega t} \times \begin{cases} e^{ik_0 z} + R_0 e^{-ik_0 z}, & \text{for } z \leq 0, \\ T e^{ikz} + R e^{-ikz}, & \text{for } 0 \leq z \leq d, \\ T_0 e^{ik_0(z-d)}, & \text{for } d \leq z. \end{cases}$$



The four unknown (complex) coefficients R , R_0 , T , and T_0 , participating in these expressions may be found by plugging them into four boundary conditions which express the continuity of the electric field E and the magnetic field $H = \pm E/Z$ at both surfaces ($z = 0$ and d). This gives us the following system of four linear equations:

$$1 + R_0 = T + R, \quad \frac{1 - R_0}{Z_0} = \frac{T - R}{Z},$$

$$T e^{ikd} + R e^{-ikd} = T_0, \quad \frac{T e^{ikd} - R e^{-ikd}}{Z} = \frac{T_0}{Z_0}.$$

Solving this system, we get

$$T_0 = \frac{4ZZ_0}{(Z + Z_0)^2 - (Z - Z_0)^2 e^{2ikd}}, \quad (*)$$

and the power transmission coefficient may now be readily calculated as $F = |T_0|^2$.

(ii) The easiest way to analyze the dependence of T_0 and F of the system parameters (at constant and positive ε and μ , and hence constant and positive Z and Z_0), is to consider the diagram below which shows the complex denominator of Eq. (*) as the difference of two vectors, of lengths $(Z + Z_0)^2$ and $(Z - Z_0)^2 < (Z + Z_0)^2$, with angle $2kd$ between them.

It is clear, first of all, that the modulus the denominator, and hence $|T_0|$ and F , oscillate as a functions of both wave frequency ω and the film thickness d , with the period $\Delta(2kd) = 2\pi$, so that the transmitted power has equal maxima at

$$kd = \pi m, \quad m = 0, 1, \dots$$

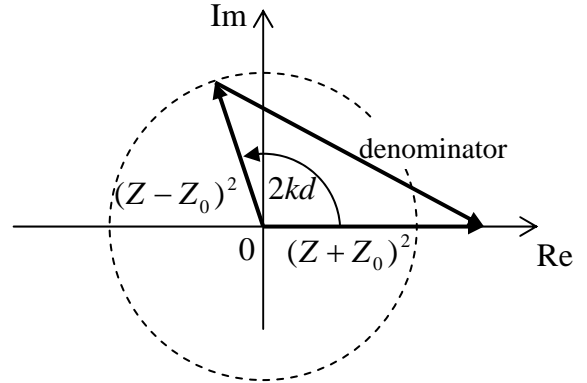
i.e. at the well-known “constructive interference” condition that the film thickness equals an integer number of half-wavelengths $\lambda/2 = \pi/k$.¹ A less trivial fact is that in each of the maxima the transmission is perfect:

$$T_{\max} = T \Big|_{kd=\pi m} = \frac{4ZZ_0}{(Z+Z_0)^2 - (Z-Z_0)^2} = 1, \quad F_{\max} = 1,$$

regardless of the Z/Z_0 ratio. The “only” feature this ratio determines is the sharpness of the resonances: the farther it deviates from 1, the lower is the transmission minimum,

$$(T_0)_{\min} = T_0 \Big|_{kd=\pi(m+1/2)} = \frac{4ZZ_0}{(Z+Z_0)^2 + (Z-Z_0)^2} = \frac{2(Z/Z_0)}{(Z/Z_0)^2 + 1}, \quad F_{\min} = \frac{4(Z/Z_0)^2}{[(Z/Z_0)^2 + 1]^2} \leq 1,$$

and the faster the power transmission drops with the deviation of d and/or k from the resonance.



Problem 1.2 (15 points). A monochromatic, plane wave is incident from inside a medium with $\varepsilon\mu > \varepsilon_0\mu_0$ on its plane surface, at an angle of incidence θ larger than the critical angle $\theta_c = \arcsin(\varepsilon_0\mu_0/\varepsilon\mu)^{1/2}$. Calculate the depth of penetration of a field induced by the wave into the free space and analyze its dependence on θ . Does the result depend on the wave polarization?

Solution: Selecting axes x and z in the plane of incidence (see Fig. on the right), we can claim that in our problem $\partial/\partial y = 0$, so that the 3D wave equation, which describes the EM fields at $z > 0$ (see, e.g, Eq. (7.3) of the lecture notes), may be reduced to the 2D form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) f = 0,$$

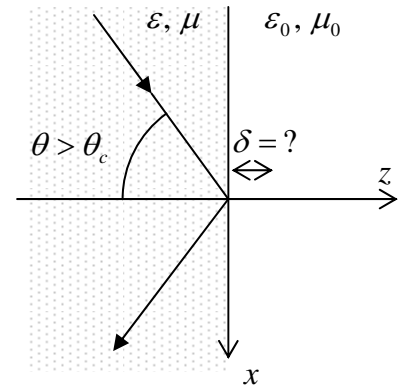
where f is any field component, and $c = (\varepsilon_0\mu_0)^{1/2}$ is the speed of light. Looking for the solution of this differential equation in the natural form

$$f = f_\omega \exp\{i[k_x x + k_z z - \omega t]\},$$

where the horizontal component k_x of the wave vector has to match those of the incident and reflected waves, $k_x = k \sin \theta = \omega(\varepsilon\mu)^{1/2} \sin \theta$, in order to satisfy the boundary conditions at $z = 0$, we get the matching condition

$$k_x^2 + k_z^2 = k_0^2, \quad (*)$$

where $k_0 \equiv \omega/c$. At $\theta > \theta_c \equiv \arcsin(k_0/k)$, we have $k_x > k_0$, so that Eq. (*) may be only satisfied by purely imaginary $k_z = i/\delta$, where δ is the penetration depth we are seeking for. As a result, Eq. (*) yields



¹ A similar effect in quantum mechanics is called *resonant tunneling*.

$$\delta = \frac{1}{(k_x^2 - k_0^2)^{1/2}} = \frac{c \sin \theta_c}{\omega(\sin^2 \theta - \sin^2 \theta_c)^{1/2}} = \frac{\lambda_0}{2\pi} \frac{\sin \theta_c}{(\sin^2 \theta - \sin^2 \theta_c)^{1/2}}.$$

We see that δ is of the order of free space wavelength $\lambda_0 = 2\pi/k_0$, but diverges as θ approaches θ_c .

Finally, notice that the result for δ has been derived by a “kinematics” analysis, and hence (just like the reflection and Snell laws) does not depend on the incident wave polarization.