PHY 505 **Classical Electrodynamics I** Fall 2010

Homework 1 with solutions

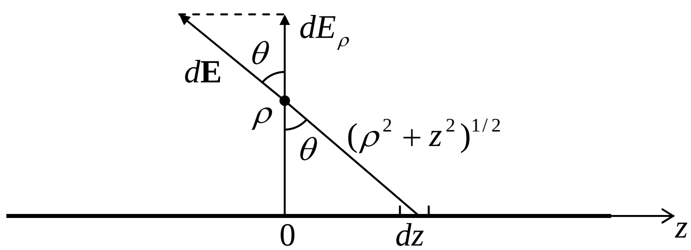
Problem 1.1 (to be graded of 10 points). Calculate the electric field created by a thin, long, straight filament, electrically charged with a constant linear density λ, using two approaches:

(i) directly from Eq. (8), and

(ii) using the Gauss law.

Solutions:

(i) From the translational and axial symmetries of the problem, it is clear that $\mathbf{E}(\mathbf{r}) = \mathbf{n}_\rho E(\rho)$, where ρ is the distance from the filament.[1] Let us select the plane of drawing so that it contains both the filament and the observation point, and take the line of filament for axis z (see Fig. below).



Then $E(\rho)$ may be calculated as

$$E = \int_{z=-\infty}^{z=+\infty} dE_\rho = \int_{z=-\infty}^{z=+\infty} dE \cos\theta = \int_{z=-\infty}^{z=+\infty} dE \frac{\rho}{\left(\rho^2 + z^2\right)^{1/2}},$$

where dE is the magnitude of the elementary contribution to the field, created by a small segment dz of the filament, with electric charge λdz. According to Eq. (1.3) of the lecture notes,

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{\lambda dz}{\rho^2 + z^2},$$

so that the total field

$$E = \frac{\lambda\rho}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{dz}{\left(\rho^2 + z^2\right)^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0 \rho} \int_{-\infty}^{+\infty} \frac{d\xi}{\left(1 + \xi^2\right)^{3/2}} = \frac{\lambda}{2\pi\varepsilon_0 \rho}. \qquad (*)$$

(For the last transition, I have used the well-known table integral – see, e.g., MA Eq. (6.5b).)

(ii) Taking a round cylinder of radius ρ and length L, with its axis on the filament, for the Gaussian volume, we ensure that the electric field E is the same, and perpendicular to the volume boundary on its side walls, while the field flux through cylinder's "lids" is zero. As a result, Eq. (1.16) yields

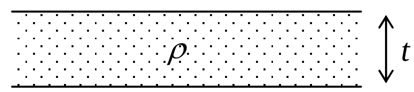
[1] I regret using the same letter (ρ) as for the charge density per unit volume (which is not used in this problem), but both notations are traditional. Let me hope this will not result in any confusion.

$$2\pi\rho LE = \frac{\lambda L}{\varepsilon_0},$$

immediately giving the same result (*).[2]

We see that for this, highly symmetric problem both solution ways are readily doable, but the Gauss method is still easier.

Problem 1.2 (10 points). Use any two (different) approaches you like to calculate the distribution of electrostatic potential ϕ and electric field **E** created, in otherwise free space, by a plane layer of thickness t, with a uniformly distributed charge of density ρ (see Fig. on the right).



Solutions:

(i) Using as the Gauss volume a pillbox similar to that discussed in the lecture notes (Fig. 1.4), for $|z| > t/2$ we get the same field as for the thin charged plane - see Eq. (24), where now $\sigma = \rho t$, so that

$$E = \frac{\rho}{\varepsilon_0}\frac{t}{2}, \quad E_z = \frac{\rho}{\varepsilon_0}\frac{t}{2}\text{sgn}(z), \quad \text{for } |z| > \frac{t}{2},$$

where z is the Cartesian coordinate perpendicular to the layer, with $z = 0$ in its middle.. On the other hand, if the pillbox thickness is smaller than the charged layer thickness, then the areal density participating in the Gauss law is only that inside the pillbox, so that $\sigma = 2|z|\rho$, so that the law yields

$$E = \frac{\rho}{\varepsilon_0}|z|, \quad E_z = \frac{\rho}{\varepsilon_0}z, \quad \text{for } |z| < \frac{t}{2}.$$

The formulas (of course) give the same result at the layer boundaries.

Now, the electrostatic potential may be obtained by integration of E, for example, along the vertical axis:

$$\phi = -\int^{z} E_z(z')dz' \int = \phi_{|z=0} - \int_0^z E(z')dz' = \phi_{|z=0} - \frac{\rho}{2\varepsilon_0}\times\begin{cases} z^2, & \text{for } |z| < t/2, \\ t\left(|z| - \frac{t}{4}\right), & \text{for } |z| > t/2, \end{cases} \qquad (*)$$

where the potential value on the symmetry plane is arbitrary. Note if the charge per unit area, $\sigma = t\rho$, is fixed, the electric field outside the layer does not depend on its thickness, but the electric potential does, reflecting the additional energy of the field inside the layer due to its final thickness.

Another notable feature of result (*) is that the electrostatic potential grows infinitely as $z \to \pm\infty$. This is of course an artifact of our (unphysical) assumption that the charged sheet's area is infinite. For any realistic (finite-area) sheet the electric field and the potential start to fall as soon as $|z|$ becomes larger than the sheet's lateral size.

(ii) The Poisson equation for this system takes the form

[2] Please notice the local (intra-problem) numbering of equations in the solutions.

$$\nabla^2 \phi = -\begin{cases} \rho/\varepsilon_0, & \text{for } |z| < t/2, \\ 0, & \text{for } |z| > t/2. \end{cases}$$

It is evidently satisfied by a 1D solution $\phi = \phi(z)$, so that the equation takes the form:

$$\frac{d^2\phi}{dz^2} = -\begin{cases} \rho/\varepsilon_0, & \text{for } |z| < t/2, \\ 0, & \text{for } |z| > t/2. \end{cases}$$

Integrating the former (top) equation once, we get

$$\frac{d\phi}{dz} = -\frac{\rho}{\varepsilon_0} z + \text{const.}$$

The integration constant may be determined from the problem symmetry: $\phi(-z) = \phi(z)$. At $z = 0$, this relation requires

$$\frac{d\phi}{dz}_{|z=0} \equiv \lim_{z\to 0} \frac{\phi(z) - \phi(-z)}{2z} = 0,$$

showing that the integration constant should equal zero. Now we can integrate the result for $d\phi/dz$ over z again, arriving at the top line of Eq. (*). The integration of the Poisson for region $|z| > d/2$ is similar, with the boundary conditions provided by the continuity of functions $\phi(z)$ and $d\phi/dz = -E_z(z)$ at the interfaces $|z| > d/2$. The result is given by the lower line of Eq. (*).

We see that the second method of solution (based on the Poisson equation) is somewhat less convenient than using the Gauss law. As a partial compensation, the intermediate result, $d\phi/dz$, already gives the electric field E_z (with the minus sign).

Problem 1.3 (10 points).Calculate:

(i) the distribution of ϕ and **E** in space, and
(ii) the electrostatic energy per unit area,

of two thin, parallel planes with equal and opposite charges of constant areal density σ, separated by distance d (see Fig. on the right).

Solution: From the similar problem solved in the lecture notes (see Fig. 1.4 and its discussion), it is clear that the electric field everywhere is vertical and constant within each of three ranges:

- below the lower plane ($E_z = E_-$),
- between the planes ($E_z = E_0$), and
- above the top plate ($E_z = E_+$).

Applying the Gauss law to three pillboxes with three possible combinations of the lid positions, we get three equations for these constants:

$$E_+ - E_0 = +\sigma/\varepsilon_0, \qquad E_0 - E_- = -\sigma/\varepsilon_0, E_+ - E_- = 0,$$

which give $E_+ = E_- = 0$, $E_0 = -\sigma/\varepsilon_0$. Thus, the field exits only between the planes, producing the electrostatic potential

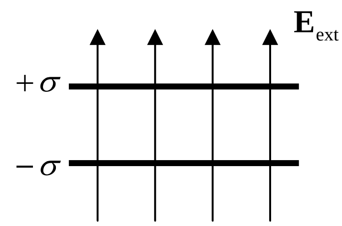
$$\phi = \phi_{|z=0} - \int_0^z E(z')dz' = \phi_{|z=0} + \frac{\sigma}{\varepsilon_0} \times \begin{cases} z, & \text{for } |z| < d/2, \\ (d/2)\,\text{sgn}(z), & \text{for } |z| > d/2, \end{cases}$$

so that the potential difference ("*voltage*") between the planes is

$$V \equiv \phi\left(+\frac{d}{2}\right) - \phi\left(-\frac{d}{2}\right) = \frac{\sigma}{\varepsilon_0} d \equiv \frac{\sigma}{C_0},$$

where $C_0 \equiv \sigma/V = \varepsilon_0/d$ is the *specific capacitance* of this simple system (which is equivalent to a plane capacitor – see Chapter 2). Now the easiest way to calculate the electrostatic energy per unit area is to use Eq. (1.53):

$$\frac{U}{A} = \frac{\varepsilon_0}{2} \int_{-d/2}^{+d/2} E^2 dz = \frac{\varepsilon_0 E_0^2}{2} d = \frac{\sigma^2}{2\varepsilon_0} d = \frac{C_0 V^2}{2}.$$



Problem 1.4 (10 points). The system analyzed in Problem 1.3 (two thin, parallel, oppositely charged planes) is now placed inside an external, uniform, normal electric field $E_{ext} = \sigma/\varepsilon_0$ – see Fig. on the right. Find the forces (per unit area) acting on each plane, by two methods:

(i) directly from the electric field distribution, and

(ii) from the potential energy of the system.

Solutions:

(i) In order to calculate force F acting on the positively charged plane, we have to neglect the field of this plane itself,[3] and add the external field directed up and equal to σ/ε_0, and the field of the negatively charged plane, directed down and equal to $\sigma/2\varepsilon_0$ – see Fig. 1.4 and Eq. (1.24) of the lecture notes. The net field equals $\sigma/2\varepsilon_0$ and is directed up, so that the force per unit area is

$$\frac{F}{A} = \frac{q}{A}\frac{\sigma}{2\varepsilon_0} = \frac{\sigma^2}{2\varepsilon_0}, \qquad (*)$$

and is directed up. A similar calculation for the negatively charged plane yields force of the same magnitude, but directed down.

(ii) Applying the Gauss law to the same 3 pillboxes as were used to solve Problem 1.3, for the full field values we get $E_+ = E_- = E_{ext} = \sigma/\varepsilon_0$, $E_0 = 0$. With these values, integral (1.67) formally diverge, even if calculated per unit area. However, since fields E_+ and E_- outside the gap between the planes do not depend on distance d between them, we may limit the integration by a pillbox of thickness $d_0 > d$, containing areas A of both planes. with both lids outside the gap. Then Eq. (1.67) yields

$$\frac{U}{A} = \frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 (d_0 - d) = \text{const} - \frac{\sigma^2}{2\varepsilon_0} d\,.$$

[3] Different elementary charges of the same plane do Coulomb-interact, but the elementary forces between them are directed along the plane, and cancel at summation, due to the axial symmetry.

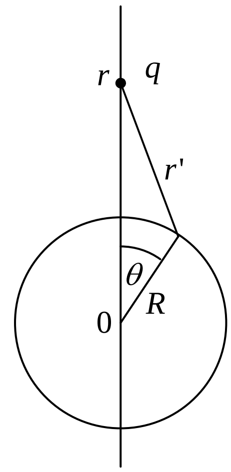
Now the vertical force (per unit area) exerted at the top plane may be calculated as $F/A = -\partial(U/A)/\partial d$, giving the same result (*) as the first method. The same is true for the lower plane whose vertical coordinate is (const – d), so that U/A should be differentiated over ($-d$) rather than d.

Problem 1.5 (10 points).

. (i) By a direct calculation, find the average electric potential of the spherical surface of radius R, created by a point charge q located at distance $r > R$ from the sphere's center.

(ii) Use the result to prove the following general *mean value theorem*: The electric potential at any point is always equal to its average value on any spherical surface with the center at that point, and containing no electric charges inside it.

Solutions:



(i) Using the evident axial symmetry of the problem (see Fig. on the right), we get:

$$\phi_{\text{ave}} \equiv \frac{1}{4\pi}\oint \phi(\theta) d\Omega = \frac{2\pi}{4\pi}\int_0^\pi \phi(\theta)\sin\theta d\theta = \frac{1}{2}\int_0^\pi \frac{q}{4\pi\varepsilon_0 r'}\sin\theta d\theta,$$

where r' is the distance between the point charge and the observation point:

$$(r')^2 = R^2 + r^2 - 2Rr\cos\theta .$$

The integral may be readily taken via the transfer to a new variable $\xi \equiv \cos\theta$ (so that $\sin\theta\, d\theta = d\xi$):

$$\phi_{\text{ave}} = \frac{1}{2}\frac{q}{4\pi\varepsilon_0}\int_1^{-1}\frac{d\xi}{\left[R^2 + r^2 - 2Rr\xi\right]^{1/2}} = \frac{1}{2}\frac{q}{4\pi\varepsilon_0 r'}\frac{2}{(-2Rr)}\left[R^2 + r^2 - 2Rr\xi\right]^{1/2}\Big|_{\xi=-1}^{\xi=+1} .$$

$$= \frac{1}{2}\frac{q}{4\pi\varepsilon_0}\frac{2}{(-2Rr)}\left\{\left[R^2 + r^2 - 2Rr\right]^{1/2} - \left[R^2 + r^2 + 2Rr\right]^{1/2}\right\} = \frac{q}{4\pi\varepsilon_0 r}.$$

We see that the average coincides with the potential value in the middle of the sphere. (Notice that this result is only valid for the case $r > R$.

(ii) The proof is elementary using the linear superposition principle: since the relation

$$\phi(r) = \frac{q}{4\pi\varepsilon_0 r} = \phi_{\text{ave}}$$

holds for each point charge located outside the sphere (as was proved in the previous problem), it is also true for any system of such charges.