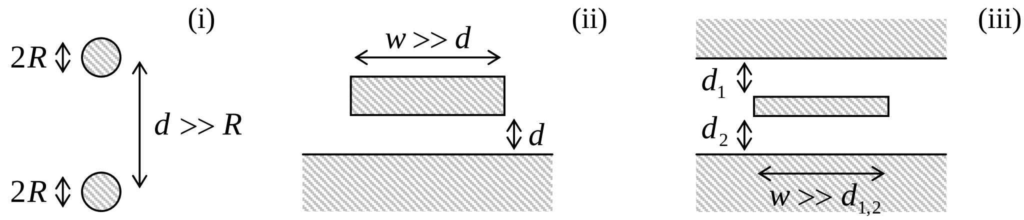
PHY 506 **Classical Electrodynamics II** Spring 2011

Homework 2 with solutions

Problem 2.1 (to be graded of 25 points). Calculate impedance Z_W of long, straight TEM transmission lines formed by metallic electrodes with the cross-sections shown in Fig. below:

(i) two round, parallel wires, separated by distance $d >> R$,
(ii) *microstrip line* of width $w >> d$,
(iii) *stripline* with $w >> d_1 \sim d_2$,

in all cases using the macroscopic boundary conditions on metallic surfaces. Assume that the conductors are embedded into a linear dielectric with constant ε and μ.



Solutions:

(i) From the solution of Problem 2.3 with $R_1 = R_2 = R$, generalized to dielectric filling by the replacement of ε_0 with ε, the mutual capacitance per unit length is

$$C_0 = \frac{\pi\varepsilon}{\ln(d/R)}.$$

The inductance L_0 per may be found either from magnetostatics (e.g., using the generalized Ampère law), or directly from Eq. (7.109):

$$L_0 = \frac{\varepsilon_0\mu_0}{C_0} = \frac{\mu_0}{\pi}\ln\frac{d}{R}.$$

Combining these expressions, we get

$$Z_W \equiv \left(\frac{L_0}{C_0}\right)^{1/2} = \frac{Z}{\pi}\ln\frac{d}{R}, \quad Z \equiv \left(\frac{\mu}{\varepsilon}\right)^{1/2}.$$

This expression is close in structure to that for the coaxial cable – see Eq. (7.113) of the lecture notes, with an extra factor of 2, due to two (rather than one) thin conductors, with distance d between the wires playing the role almost similar to the diameter of the outer tube of the cable.

Practical notice: this system is very important for modern information technology. Pairs of parallel copper wires (typically spaced by distance $d \sim 4R$ and twisted into a spiral to avoid picking up and creating interferences) are broadly used for carrying phone and internet signals (for example, in the Ethernet and DSL technologies) with frequencies up to a few hundred GHz.

(ii) Neglecting the fringe field effects (whose relative contribution scales as $d/w << 1$), we may use Eq. (3.36) for the capacitance per unit length:

$$C_0 = \frac{\varepsilon w}{d}.$$

The inductance per unit length may be, again, found either from magnetostatics (see, e.g., the solution of Problem 5.2) or directly from the universal Eq. (7.110) of the lecture notes:

$$L_0 = \frac{\mu d}{w}.$$

Combining these expressions, we get

$$Z_W = Z\frac{d}{w} << Z .$$

Such low-impedance lines are broadly used in electronics, e.g., on simple (single-layer) printed circuit boards. Due to the low fringe fields, they provide low mutual interference even if spaced closely (at distances ~ w).

(iii) The stripline geometry, which is broadly used in multi-layer printed circuit boards and for on-chip interconnects in microelectronics, provides even lower interferences between adjacent transmission lines. It parameters may be readily found, considering C_0 as a parallel connection of two plane capacitors:

$$C_0 = \varepsilon w\left(\frac{1}{d_1}+\frac{1}{d_2}\right), \quad L_0 = \frac{\mu}{w}\left(\frac{1}{d_1}+\frac{1}{d_2}\right)^{-1}, \quad Z_W = Z\frac{1}{w}\left(\frac{1}{d_1}+\frac{1}{d_2}\right)^{-1} = Z\frac{d_1 d_2}{w(d_1+d_2)}.$$

Problem 2.2 (15 points). Present the H_{10} wave in a rectangular waveguide (Fig. 7.21 of the lecture notes) as a sum of two plane waves, and discuss the physics behind such presentation.

Solution: Let us use Eq. (7.134) of the lecture notes to write the expression for the instant electric field of the wave:

$$\mathbf{E}(\mathbf{r},t) = \text{Re}\left[i\frac{ka}{\pi}ZH_l \sin\frac{\pi x}{a}e^{i(k_z z-\omega t)}\right]\mathbf{n}_y = \text{Re}\left[E_\omega e^{i(\frac{\pi}{a}x+k_z z-\omega t)} - E_\omega e^{i(-\frac{\pi}{a}x+k_z z-\omega t)}\right]\mathbf{n}_y, \qquad (*)$$

with $E_\omega = (ka/2\pi)ZH_l$. This is nothing more that the sum of two plane waves, linearly polarized along axis y, with equal amplitudes E_ω, which propagate along wave vectors $\mathbf{k}_\pm$ with Cartesian components $k_x = \pm\pi/a$, $k_y = 0$, and k_z defined by Eqs. (7.98b) and (7.128) (with $n = 1$, $m = 0$), so that

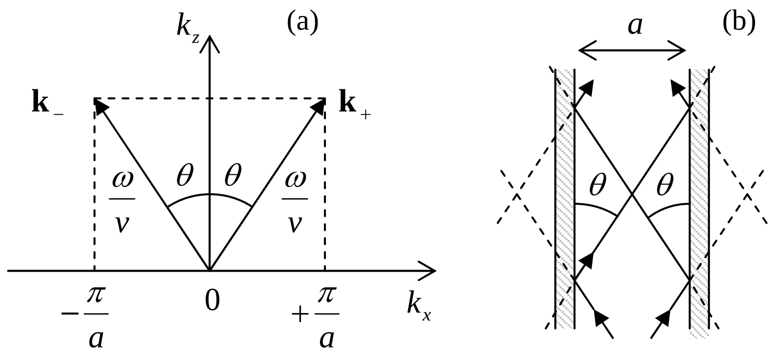
$$k_x^2 + k_y^2 + k_z^2 = k_\pm^2 = k^2 = \frac{\omega^2}{v^2},$$

- see panel (a) in Fig. below. This figure shows that the plane wave propagation angle θ satisfies the following relation:

$$\sin\theta = \frac{|k_x|}{k} = \frac{\pi}{ka} = \frac{\lambda_0/2}{a},$$

where $\lambda_0 = 2\pi/k$ is the TEM wavelength. This angle asymptotically approaches zero at very high frequencies ($\omega >> \omega_c$), but grows when the frequency is reduced toward the cutoff value ω_c, so that at ω

$\to \omega_c$, $\theta \to \pi/2$ ($k_z \to 0$). Notice that the cutoff frequency corresponds to the half-wave being exactly equal to the wide side of the waveguide, so that longer waves cannot propagate in it.



The fact that we are dealing with the usual plane waves may be confirmed by the field amplitude ratio calculation. Indeed, the similar decomposition of the H_{10} mode's magnetic field given by Eqs. (7.127) and (7.133), yields two waves:

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{2} H_l \,\mathrm{Re}\left[\left(-\frac{k_z a}{\pi}\mathbf{n}_x + \mathbf{n}_z\right) e^{i(\frac{\pi}{a}x + k_z z - \omega t)} + \left(\frac{k_z a}{\pi}\mathbf{n}_x + \mathbf{n}_z\right) e^{i(-\frac{\pi}{a}x + k_z z - \omega t)}\right], \qquad (**)$$

each with amplitude,

$$|H_\omega| = \frac{H_l}{2}\left[\left(\frac{k_z a}{\pi}\right)^2 + 1\right]^{1/2} = H_l \frac{ka}{2\pi} = \frac{1}{Z}|E_\omega| \ ,$$

whose ratio to E_ω is given by the plane wave impedance $Z = (\mu/\varepsilon)^{1/2}$, regardless of the ratio ω/ω_c.

Now, let us discuss why such deconstruction of waves in waveguide into plane waves is possible. If these two plane waves propagated in unlimited isotropic media, we could notice, first of all, that insertion of metallic walls normal to axis y at any locations (say, at $y = 0$ and $y = b$, see Fig. 7.21) would not perturb the field between them. Indeed, as we know from Chapter 2, the electric field perpendicular to the conductor surface is screened from penetration into the conductor's bulk by surface charges with density $\sigma(x,y) = \varepsilon E_z(x,y)$, without any perturbation of the applied field. Similarly, the magnetic fields of these two plane waves are tangential to the wall surfaces, and is shielded from penetration inside the walls by the skin-effect currents, without field perturbation outside the walls.

The situation with the "vertical" walls ($x = 0$ and $x = a$, see Fig. 7.21) is more involved, since in arbitrary location of such a wall, the wave sum would have some tangential component of the electric field, and normal component of the magnetic field, which cannot interact with the wall without incident wave perturbation. However, as Eqs. (*) and (**) show, at positions $k_x x_n = n\pi$, with any integer n, the sums of the components equal zero, and the wall insertion leaves the field between the walls intact.

One may also say that the H_{10} wave is formed by one TEM wave repeatedly reflected from the "vertical" walls of the waveguide – see panel (b) of the Fig. above).