PHY 505 **Classical Electrodynamics I** Fall 2010

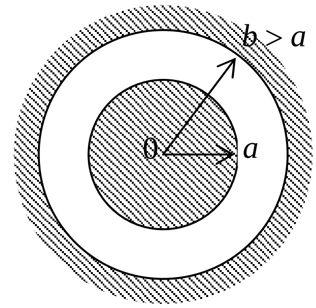
Homework 2 with solutions

Problem 2.1 (to be graded of 10 points). Use the Gauss law to calculate the mutual capacitance of the following 2-electrode systems, with the same cross-section (see Fig. on the right):

(i) a conducting sphere inside a concentric spherical cavity in another conductor, and

(ii) a conducting cylinder inside a coaxial cavity in another conductor. (In this case, we speak about capacitance per unit length.)

Compare the results with those obtained in class using the Laplace equation solution.



Solutions:

(i) Applying the Gauss Law to a sphere of radius r in the range $a < r < b$, for the electric field $\mathbf{E} = E(r)\mathbf{n}_r$, we get

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2},$$

where Q is the charge of the inner conductor. Integrating this result, we find that voltage $V \equiv \phi(b) - \phi(a)$ between the electrodes is

$$V = \int_a^b E(r)dr = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right),$$

so that the mutual capacitance

$$C_m \equiv \frac{Q}{V} = 4\pi\varepsilon_0\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} = 4\pi\varepsilon_0 \frac{ab}{b-a},$$

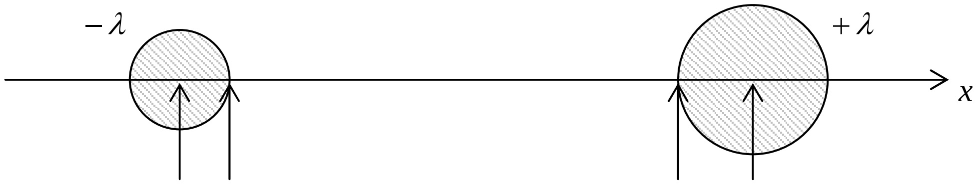
in agreement with Eq. (2.56) of the lecture notes.

(ii) An absolutely similar calculation, but applied to a cylinder, with charge $\lambda \equiv Q/L$ per unit length, gives

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}, \qquad \frac{C_m}{L} = \frac{2\pi\varepsilon_0}{\ln(b/a)},$$

in agreement with Eq. (2.49).

Problem 2.2 (10 points). Following the class discussion of two weakly coupled conducting spheres, find an approximate expression for the mutual capacitance (per unit length) between two thin, parallel wires, each with a round cross-section, but its own diameter. Both diameters, D_1 and D_2, are much smaller than the distance d between the wires. Compare the result with that for two spheres, and interpret the difference.



Solution: This problem is close to that solved in class for two small spheres, but we should be more accurate with the selection of the additive constant in the electrostatic potential, because the potential of a single, uniformly charged wire is proportional to ln r and hence diverges at $r \to \infty$. As a result, the *self-capacitance* of a single wire of an infinite length is not well defined – see, e.g., the discussion accompanying Eq. (2.50) of the lecture notes. This is why it is better to immediately proceed to the calculation of the *mutual* capacitance between two wires by considering them uniformly charged with equal but opposite linear densities $\pm\lambda$ - see Fig. above.

Just as in the problem solved in class, when calculating the electric field E_1 created by the left wire (charge $-\lambda$) alone, we can neglect the effects of the second wire. Applying the Gauss Law to a round cylinder of radius r, coaxial with the first (left) wire, we get (just like in Problem 1 above):

$$E_{1r} = -\frac{\lambda}{2\pi\varepsilon_0 r},$$

so that the electric field along the line connecting the wire centers is

$$E_{1x} = -\frac{\lambda}{2\pi\varepsilon_0 x}.$$

Similarly, with our choice of the origin of axis x (see Fig. above), the field created on that line by the second wire is

$$E_{2x} = -\frac{\lambda}{2\pi\varepsilon_0 (d-x)}.$$

(Here we took into account the opposite sign of the distributed charge.)

Now we can get calculate voltage V between the wires by integration of the total field $E_x = E_{1x} + E_{2x}$ along axis x:

$$V = \phi\Big|_{x=d-D_2/2} - \phi\Big|_{x=D_1/2} = -\int_{D_1/2}^{d-D_2/2} (E_{1x} + E_{2x}) dx = \frac{\lambda}{2\pi\varepsilon_0} \int_{D_1/2}^{d-D_2/2} \left(\frac{1}{x} + \frac{1}{d-x}\right) dx$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\frac{d-D_2/2}{D_1/2} - \ln\frac{D_2/2}{d-D_1/2}\right) \approx \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{4d^2}{D_1 D_2},$$

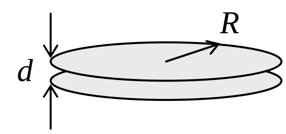
where the last transition is based on the strong inequality $d >> D_{1,2}$. Hence the mutual capacitance per unit length

$$\frac{C_m}{L} \equiv \frac{\lambda}{V} \approx \frac{2\pi\varepsilon_0}{\ln\left(\dfrac{4d^2}{D_1 D_2}\right)} = \frac{\pi\varepsilon_0}{\ln\left(d / R_{\text{mean}}\right)}, \quad \text{where } R_{\text{mean}}^2 \equiv \frac{D_1}{2}\frac{D_2}{2}.$$

This result shows, that, in contrast with the similar problem for spheres, the mutual capacitance does depend on the distance between the conductors (decreasing with d), albeit very weakly (logarithmically). This a result of the logarithmic divergence of the single wire's potential, mentioned above. Also notice the close similarity between the last formula and that for the "coaxial cable" geometry considered in Sec. 2.3.

Problem 2.3 (20 points). Using the results for a single thin, round, conducting disk, obtained in class, consider a system of two such disks at a small distance $d << R$ from each other - see Fig. on the right. In particular, calculate:

(i) the reciprocal capacitance matrix of the system,
(ii) the mutual capacitance between the disks,
(iii) the partial capacitance, and
(iv) the effective capacitance of one disk,



(all in the first non-vanishing approximations in $d/R << 1$). Compare the results (ii)-(iv) and interpret their similarities and differences.

Solutions:

(i) Due to the symmetry of this system, and the general reciprocity property given by Eq. (2.22) of the lecture notes, the general linear relations (2.19) are reduced to

$$\begin{aligned} \phi_1 &= pQ_1 + p'Q_2, \\ \phi_2 &= p'Q_1 + pQ_2. \end{aligned} \qquad (*)$$

At $d << R$, approximate formulas for the remaining two coefficients $p = p_{11} = p_{22}$ and $p' = p_{12} = p_{21}$ may be found from the two problems already solved in class.

Indeed, if the both disks carry equal charges, $Q_1 = Q_2 = Q/2$ (of the same sign), there is virtually no field inside the gap separating them, while the field outside is the almost the same as that of a single disk charged by charge Q. According to Eq. (2.68), in this case

$$\phi_1 = \phi_2 = \frac{Q}{8\varepsilon_0 R}.$$

On the other hand, for this case Eqs. (*) are reduced to

$$\phi_1 = \phi_2 = \frac{p + p'}{2} Q.$$

Comparing these two results, we get

$$p + p' = \frac{1}{4\varepsilon_0 R}. \qquad (**)$$

On the contrary, if the disk charges are equal but opposite, say $Q_1 = -Q_2 = Q$, then the system is noting more than a plane capacitor (with a strong, uniform field inside the gap and a negligible field outside it), so that according to Eq. (2.28),

$$\phi_1 = -\phi_2 = \frac{V}{2} = \frac{Q}{2C_m} = \frac{Qd}{2\pi\varepsilon_0 R^2}.$$

Comparing this result with the prediction of Eqs. (*) for this case,

$$\phi_1 = -\phi_2 = (p - p')Q,$$

we get one more equation for p and p':

$$p - p' = \frac{d}{2\pi\varepsilon_0 R^2} << p + p'. \qquad (***)$$

Now solving the system of two linear equations (**) and (***), we get

$$p = \frac{1}{8\varepsilon_0 R} + \frac{d}{4\pi\varepsilon_0 R^2} = \frac{1}{8\varepsilon_0 R}\left(1 + \frac{2d}{\pi R}\right), \quad p' = \frac{1}{8\varepsilon_0 R} - \frac{d}{4\pi\varepsilon_0 R^2} = \frac{1}{8\varepsilon_0 R}\left(1 - \frac{2d}{\pi R}\right).$$

(ii) The *mutual* capacitance, defined by Eq. (2.26),

$$C_m = \frac{1}{(p_{11} + p_{22}) - (p_{12} + p_{21})} = \frac{1}{2(p - p')},$$

has essentially been found above – see Eq. (***), and is just the planar capacitance (2.28):[1]

$$C_m = \frac{\pi\varepsilon_0 R^2}{d}.$$

(iii) The *partial* capacitances of the disks are equal, $C_1 = C_2 = 1/p_{11} = 1/p_{22} = 1/p$, and in the first approximation are independent of d:

$$C_1 = C_2 \approx 8\varepsilon_0 R.$$

(iv) On the opposite, the *effective* capacitances of the disks, defined by Eq. (2.34), are much larger and inversely proportional to the gap width:

$$C_{1,2}^{\text{ef}} \approx \frac{\pi\varepsilon_0 R^2}{d} = C_m >> C_{1,2}.$$

This (large) difference between the partial and effective capacitance (typical for all strongly-coupled systems) is very natural. If one disk is charged and the second one is not, the latter disk acquires the potential of the charged counterpart,[2] and virtually does not affect the field distribution in space. However, if the second disk is grounded, it kills the electric field everywhere besides the narrow gap between the charged disk and the "ground".

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[1] Notice that if we tried to calculate C_m from p and p', keeping only the leading terms in those coefficients, we would get an (unphysical) infinity.

[2] Electronic engineers call such electrodes "floating".