

**Problem 3.1** (to be graded of 20 points). Use the recipe outlined in Sec. 7.8 of the lecture notes to derive the characteristic equation (7.157),

$$\left( \frac{k_-^2 J_n'}{k_t J_n} + \frac{k_+^2 K_n'}{\kappa_t K_n} \right) \left( \frac{1}{k_t J_n} + \frac{1}{\kappa_t K_n} \right) = \frac{n^2}{R^2} \left( \frac{k_-^2}{k_t^2} + \frac{k_+^2}{\kappa_t^2} \right) \left( \frac{1}{k_t^2} + \frac{1}{\kappa_t^2} \right),$$

for the *HE* and *EH* modes in a round, step-index optical fiber.

*Solution:* Let us start with calculating the constant coefficient in Eq. (7.156) from the requirement that at the interface ( $\rho = R$ ), both  $E_z$  and  $H_z$  are continuous (as tangential field components), i.e.  $f_+ = f_-$ . From here and Eq. (152) we get

$$f_+ = f_- \frac{J_n(k_t R)}{K_n(\kappa_t R)} K_n(\kappa_t \rho) e^{in\varphi}.$$

Now we can use the first of Eqs. (117), together with the general vector-algebra expressions for gradient's components in polar coordinates,<sup>1</sup> which give

$$(\mathbf{n}_z \times \nabla_t f)_\rho = \frac{1}{\rho} \frac{\partial f}{\partial \varphi}, \quad (\mathbf{n}_z \times \nabla_t f)_\varphi = -\frac{\partial f}{\partial \rho},$$

to obtain the following expressions for the transversal components of the electric field:

$$\begin{aligned} E_\rho^- &= \frac{i}{k_t^2} \left[ k_z k_t J_n'(k_t \rho) E_l - k_- Z_- \frac{in}{\rho} J_n(k_t \rho) H_l \right] e^{in\varphi}, \\ E_\varphi^- &= \frac{i}{k_t^2} \left[ k_z \frac{in}{\rho} J_n(k_t \rho) E_l + k_- Z_- k_t J_n'(k_t \rho) H_l \right] e^{in\varphi}, \\ E_\rho^+ &= -\frac{i}{\kappa_t^2} \frac{J_n(k_t R)}{K_n(\kappa_t R)} \left[ k_z \kappa_t K_n'(\kappa_t \rho) E_l - k_+ Z_+ \frac{in}{\rho} K_n(\kappa_t \rho) H_l \right] e^{in\varphi}, \\ E_\varphi^+ &= -\frac{i}{\kappa_t^2} \frac{J_n(k_t R)}{K_n(\kappa_t R)} \left[ k_z \frac{in}{\rho} K_n(\kappa_t \rho) E_l + k_+ Z_+ \kappa_t K_n'(\kappa_t \rho) H_l \right] e^{in\varphi}. \end{aligned}$$

where prime means differentiation of the function over its whole argument.

Now we should write the boundary conditions for the electric field components:

$$\varepsilon_- E_\rho^- = \varepsilon_+ E_\rho^+, \quad E_\varphi^- = E_\varphi^+, \quad \text{at } \rho = R.$$

Requiring that these conditions are satisfied at all angles  $\varphi$ , we get two linear equations for the complex constants  $E_l$  and  $H_l$ . At  $\mu = \mu_+ = \mu_0$ , the boundary conditions for the transversal components of the magnetic field would give an equivalent system. The same relation ensures that the ratio  $\varepsilon_+/\varepsilon_-$  may be presented as  $k_+^2/k_-^2$ , and that  $k_+ Z_+ = k_- Z_- = \omega \mu_0$ , so that the system of equations reduces to

<sup>1</sup> See, e.g., MA Eq. (10.8).

$$k_z \left( \frac{k_-^2 J'_n}{k_t J_n} + \frac{k_+^2 K'_n}{\kappa_t K_n} \right) E_l - \frac{in}{R} \omega \mu_0 \left( \frac{k_-^2}{k_t^2} + \frac{k_+^2}{\kappa_t^2} \right) H_l = 0,$$

$$k_z \frac{in}{R} \left( \frac{1}{k_t^2} + \frac{1}{\kappa_t^2} \right) E_l + \omega \mu_0 \left( \frac{1}{k_t} \frac{J'_n}{J_n} + \frac{1}{\kappa_t} \frac{K'_n}{K_n} \right) H_l = 0.$$

Now requiring the determinant of this system to be zero (in order for them to be consistent), we arrive at the characteristic equation (7.157).

**Problem 3.2** (20 points). Find the lowest eigenfrequencies, and corresponding oscillation modes, of a round cylindrical resonator (see Fig. on the right) with perfectly conducting walls.

*Solution:* Due to the simple geometry of the system, and hence the simple structure of the resonance fields in it, we can use the first method of eigenfrequency calculation, using the analysis of the circular metallic waveguides, carried out in Sec. 7.7 of the lecture notes. In particular, we can use Eqs. (7.117) with  $E_z = 0$ , and an evident generalization of Eq. (7.140),

$$H_z = H_l J_n \left( \xi'_{nm} \frac{\rho}{R} \right) e^{in\varphi},$$

to calculate the transversal fields  $\mathbf{E}_t$  and  $\mathbf{H}_t$  in the waveguide. Without writing down these formulas explicitly, we may notice that according to Eq. (117),  $\mathbf{E}_t$  has the same phase as  $H_z$ , which does not depend on the point position on the cross-section  $z = \text{const}$ . Hence if two metallic walls, parallel to axis  $z$ , are inserted into the waveguide at distance  $\Delta z = p(\lambda_z/2) = \pi p/k_z$ , where  $p = 1, 2, \dots$ , they do not disturb the field distribution (besides turning the traveling wave into the standing one). Now requiring  $\Delta z$  to be equal  $l$ , and using the general Eq. (7.116),

$$k^2 \equiv \frac{\omega^2}{v^2} = k_t^2 + k_z^2,$$

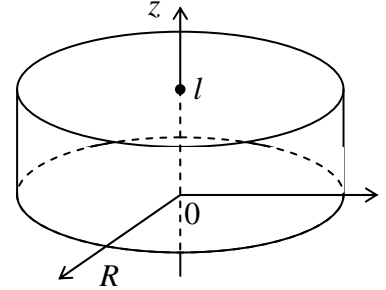
and Eq. (7.140) for eigenvalues of the transversal component  $k_t$  of the wave vector, we get the eigenfrequency spectrum:

$$\omega_{mp}^2 = v^2 k^2 = v^2 \left[ \left( \frac{\xi'_{nm}}{R} \right)^2 + \left( \frac{\pi p}{l} \right)^2 \right], \quad (*)$$

Since integers  $m$  and  $p$  have to start with 1, according to Table 7.1, the lowest of these frequencies is

$$\omega_{011}^2 = v^2 \left[ \left( \frac{\xi'_{11}}{R} \right)^2 + \left( \frac{\pi}{l} \right)^2 \right] \approx v^2 \left[ \left( \frac{1.841}{R} \right)^2 + \left( \frac{\pi}{l} \right)^2 \right]. \quad (**)$$

However, this is not necessarily the lowest frequency of the resonator. Indeed, using the absolutely similar analysis of the  $E$  waves, whose longitudinal electric field is described by the natural generalization of Eq. (7.143),



$$E_z = E_l J_n \left( \xi_{nm} \frac{\rho}{R} \right) e^{in\varphi},$$

for the corresponding modes of the resonator we get an expression very similar to Eq. (\*):

$$\omega_{nmp}^2 = v^2 \left[ \left( \frac{\xi_{nm}}{R} \right)^2 + \left( \frac{\pi p}{l} \right)^2 \right].$$

Note, however, that for the  $E$  waves, boundary conditions at the lid surfaces ( $\mathbf{E}_t = 0$ ,  $\partial H_z / \partial z = 0$ ) allow nonvanishing fields even if  $p = 0$ . (In this case neither component of  $\mathbf{E}$  or  $\mathbf{H}$  depend on  $z$ .) Hence the lowest eigenfrequency of these modes

$$\omega_{010}^2 = v^2 \left( \frac{\xi_{01}}{R} \right)^2 \approx v^2 \left( \frac{2.405}{R} \right)^2, \quad (***)$$

independently of  $l$ . Note that the field distribution in this  $E_{010}$  mode is very similar to that of the rectangular resonator – see Fig. 7.30, where  $b$  should be replaced with  $l$ , and  $a$  with  $2R$ .

The direct comparison of Eqs. (\*\*) and (\*\*\*) shows that the latter frequency is lower, and hence  $E_{010}$  is the fundamental mode of the resonator, if  $l < \pi R / (\xi_{01}^2 - \xi_{11}^2)^{1/2} \approx 2.03R$ . (A similar crossover in the rectangular resonator, disguised by the similarity of field distribution along its sides, takes place at  $l = \min[a, b]$ , i.e. essentially at the same condition as in the circular resonator, if we identify its diameter  $2R$  with the lateral size.) In the opposite case,  $H_{011}$  is the basic mode.