PHY 505 **Classical Electrodynamics I** Fall 2010

Homework 3 with solutions

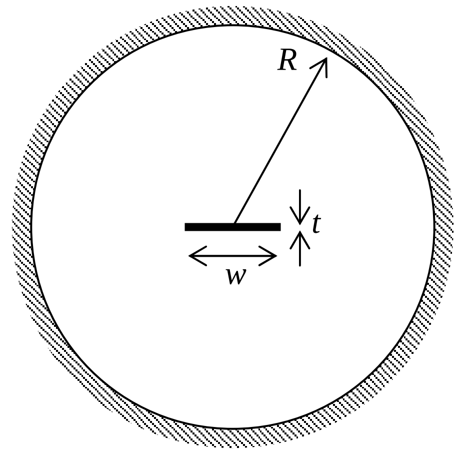
Problem 3.1 (to be graded of 15 points). Calculate the mutual capacitance (per unit length) between two cylindrical conductors, forming a system with the cross-section shown in Fig. on the right, in the limit $t << w << R$.

Hint: You may like to use the "*elliptical*" (not "ellipsoidal"!) coordinates defined by the following equation:

$$x + iy = c \times \cosh(\alpha + i\beta), \qquad (*)$$

with the appropriate choice of constant c. In these orthogonal 2D coordinates, the Laplace operator is very simple:

$$\nabla_2^2 = \frac{1}{c^2(\cosh^2\alpha - \cos^2\beta)}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right). \qquad (**)$$



(This is not quite surprising, because Eq. (*) may be also considered as a conformal map $z = c \cosh w$, where $z = x + iy$, and $w = \alpha + i\beta$.)

Solution: On the $[x, y]$ plane, the lines of constant α are ellipses with horizontal and vertical semi-axes $c\cosh\alpha$ and $c\sinh\alpha$, respectively. For $\alpha = 0$, the ellipse degenerates into a straight horizontal segment $-c < x < +c$, while for $\alpha >> 1$, the ellipse is virtually a circle of radius $\rho = (c/2)\exp\alpha$. As a result, if we select the axes x and y as shown in figure, and take $c = w/2$, the boundary conditions on the conductor surfaces (see Fig.) may be satisfied, at $t << w << R$, by a potential distribution $\phi(\alpha)$, with no dependence on β:

$$\phi(0) = 0, \quad \phi(\ln\frac{4R}{W}) = -V, \qquad (***)$$

where V is the voltage between the conductors. Hence the boundary problem may be satisfied by a function $\phi(\alpha)$ if it satisfies the 1D Laplace equation following from Eq. (**):

$$\frac{d^2\phi}{d\alpha^2} = 0.$$

This equation shows that $\phi(\alpha)$ is just a linear function $c_1\alpha + c_2$. Selecting two constants $c_{1,2}$ to satisfy boundary conditions (***), we get

$$\phi = -V\frac{\alpha}{\ln(4R/w)}.$$

To calculate the surface charge of the central conductor, we need to calculate the (normal) electric field on its surface:

$$E_n = -\frac{\partial\phi}{\partial y}\bigg|_{\substack{y=+0 \\ -w/2<x<w/2}} = \frac{V}{\ln(4R/w)(w/2)\sin\beta}\frac{\partial\alpha}{\partial(\sinh\alpha)}\bigg|_{\alpha=0} = -\frac{V}{\ln(4R/w)(w/2)\sin\beta},$$

so that the two-surface charge density

$$\sigma = 2\varepsilon_0 E_n = 2\varepsilon_0 \frac{V}{\ln(4R/w)(w/2)\sin\beta} = 2\varepsilon_0 \frac{V}{(w/2)\ln(4R/w)\left[1-(2x/w)^2\right]^{1/2}}.$$

Integrating this distribution, we get the total charge (per unit length)

$$\frac{Q}{L} = \int_{-w/2}^{+w/2} \sigma dx = \frac{2\varepsilon_0 V}{(w/2)\ln(4R/w)} \int_{-w/2}^{+w/2} \frac{dx}{\left[1-(2x/w)^2\right]^{1/2}} = \frac{2\pi\varepsilon_0}{\ln(4R/w)} V,$$

so that the mutual capacitance per unit length is

$$\frac{C_m}{L} \equiv \frac{Q/L}{V} = \frac{2\pi\varepsilon_0}{\ln(4R/w)}.$$

Comparing this result with Eq. (2.49), we see that the only difference between the capacitance of this system that of the round coaxial cable is a different factor under the logarithm.

Problem 3.2 (15 points). Formulate the 2D electrostatic problems which can be solved using each of the following analytical functions of the complex variable $z \equiv x + iy$:

(i) $w = \ln z$,

(ii) $w = z^{1/2}$.

Solve one (any) of these problems.

Solutions: For a 2D boundary problem of electrostatics to be readily solvable with a conformal mapping, the borders of the cross-section of conductors should coincide with lines of constant u or v.

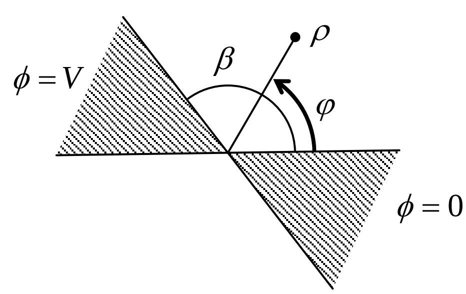
(i) Plugging relations $w = u + iv$ and $z = x + iy$ into $w = \ln z$, and equating their real and imaginary parts, we get

$$u = \frac{1}{2}\ln\left(x^2 + y^2\right), \quad v = \text{Arctan}\frac{y}{x}.$$

This means that on the [x, y] plane, lines of constant u are concentric circles, while those of equal v are straight rays going from the origin to infinity. Hence for this conformal map, two examples of easily solvable boundary problems are:

a. field distribution between two concentric cylinders of radii a and b, held at different potentials (which is exactly the Homework Problem 2.1(ii), easily solvable using the Gauss law); and

b. same for two cylindrical edges, almost touching (but still electrically insulated!) at origin – see, e.g, Fig. below.



The solution of problem a is easy: since the analytical function $u(x,y)$ satisfies the Laplace equation, and the conductors' surfaces are equipotential, then any line of constant u is equipotential one. Moreover, since the Laplace equation is reduced to $d^2\phi/d^2u = 0$, its solution is a linear function, i.e.

$$\phi = c_1 u + c_2 = c_1 \ln\sqrt{x^2 + y^2} + c_2 = c_1 \ln \rho + c_2,$$

where constants $c_{1,2}$ may be readily calculated from the potentials fixed on the concentric conducting cylinders. This is of course the result which we have already obtained by other methods.

For problem b,[1] the similar argumentation yields the equation,

$$\phi = c_1 v + c_2 = c_1 \text{Arctan}\frac{y}{x} + c_2 = c_1\varphi + c_2,$$

which is valid in free space between the conductor edges. In the simple case shown in Fig. above, this gives

$$\phi = V\frac{\varphi}{\beta}.$$

Note that according to this formula:

- the electric field lines are just concentric circles;
- the electric field on the conductor surface

$$E_n = -\frac{\partial\phi}{\partial r_n} = --\frac{\partial\phi}{\partial(\rho\varphi)}\Big|_{\rho=\text{const}} = \frac{V}{\rho\beta}$$

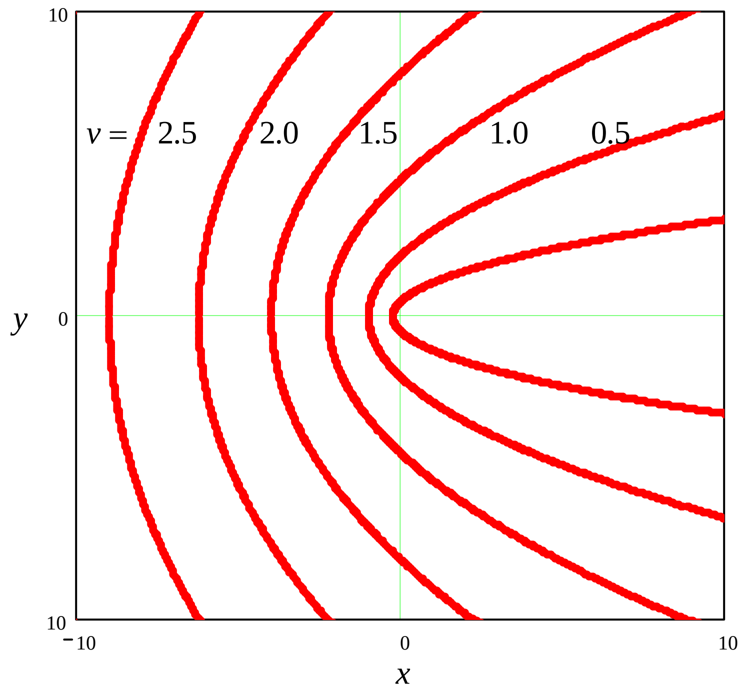
has a strong (non-integrable) divergence at $\rho \to 0$ different from the one discussed in class for 2D corners – see Eq. (2.123) of the lecture notes. The difference is due to the fact that in the latter case the potential on both sides of the corner (or edge) was the same, and the field was applied by outside sources.

(ii) Separating the real and imaginary parts of the function $\boldsymbol{w} = \boldsymbol{z}^{1/2}$, we get

$$u = \pm\left\{\frac{1}{2}\left[\sqrt{x^2 + y^2} + x\right]\right\}^{1/2}, \quad v = \pm\left\{\frac{1}{2}\left[\sqrt{x^2 + y^2} - x\right]\right\}^{1/2}, \qquad (*)$$

where the signs in both formulas should be changed simultaneously. (As is evident from Fig. 2.7b of the lecture notes, which shows the reciprocal map $\boldsymbol{z} = \boldsymbol{w}^{1/2}$, each point $\{x, y\}$ is mapped on two points $\{u, v\}$.) Figure below shows the lines of constant v on the $[x, y]$ plane; they form a set of parabolas,[2] at $v \to 0$ giving a straight segment $0 < x < +\infty$.

[1] It is probably the instructor's fate :-(to solve all the formulated problems rather than one of them.
[2] The lines of constant u are similar, with the parabolas "looking in" the opposite direction.



As a result, this mapping may be used, for example, to find the field distribution around the edge of a thin conducting sheet in the form of half-plane $x > 0$. For this problem, $d^2\phi/dv^2 = 0$, with the evident solution

$$\phi = c_1 v + c_2 = c_1 \left\{ \frac{1}{2}\left[\sqrt{x^2+y^2} - x\right]\right\}^{1/2} + c_2 .$$

The fixed potential of the conducting half-plane (say, $\phi = 0$) corresponds to $v = 0$, i.e. gives $c_2 = 0$, so that finally

$$\phi = c_1 \left\{ \frac{1}{2}\left[\sqrt{x^2+y^2} - x\right]\right\}^{1/2},$$

where constant c_1 is determined by the potential at the (distant) external electrodes creating the electric field. Near the surface of the half-plane conductor ($x > 0$, $y << x$), this expression yields

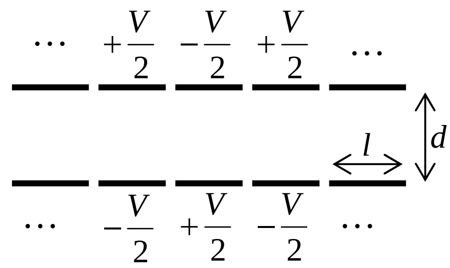
$$\phi = c_1 \frac{|y|}{2x},$$

so that the electric field at the sheet surfaces,

$$E_n = -\frac{\partial \phi}{\partial y}\Big|_{y=0} = \pm \frac{c_1}{2x},$$

has the same strong, non-integrable divergence at the edge, as in problem (i)b discussed above.

Problem 3.3 (20 points). Each electrode of a large plane capacitor is cut into long strips of equal width l, with very narrow gaps between them. These strips are kept at the alternating potentials, as shown in Fig. on the right. Use the variable separation method to calculate the electrostatic potential distribution inside the capacitor. Explore the limit $l << d$.



Solution: Due to the symmetry of the problem, electrostatic potential should vanish at:

(i) the horizontal symmetry plane passing in the middle between the planes, and
(ii) any vertical plane passing between the strips.

Selecting these planes (or rather their traces on the plane of the paper) for the coordinate axes, we see that the both functions $X(x)$ and $Y(y)$, forming any partial solution $\phi_k = XY$, should be antisymmetric. Since functions $X(x)$ should be periodic, with period $\Delta x = 2l$, they may be taken in the form of $\sin\alpha x$ with $\alpha = \pi n/l$. Hence, in order to satisfy the Laplace equation, functions $Y(y)$ have to be $\sinh \alpha y$, so that the full solution takes the form

$$\phi(x,y) = \sum_{n=1}^{\infty} \phi_n \sin\frac{\pi x n}{l} \sinh\frac{\pi n y}{l}. \qquad (*)$$

Coefficients ϕ_n should be found from the boundary conditions on the electrode surface. Since Eq. (*) already ensures the proper symmetry and periodicity of the solution, it is sufficient to require that it fits the boundary value on just one strip (say, $0 < x < l$, at $y = +d/2$):

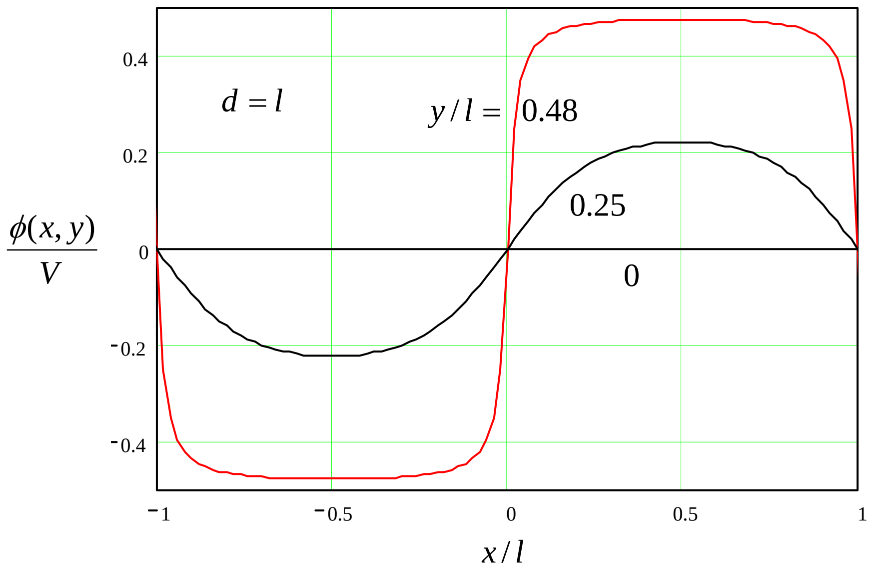
$$\frac{V}{2} = \sum_{n=1}^{\infty} \phi_n \sin\frac{\pi x n}{l} \sinh\frac{\pi n d}{2l}.$$

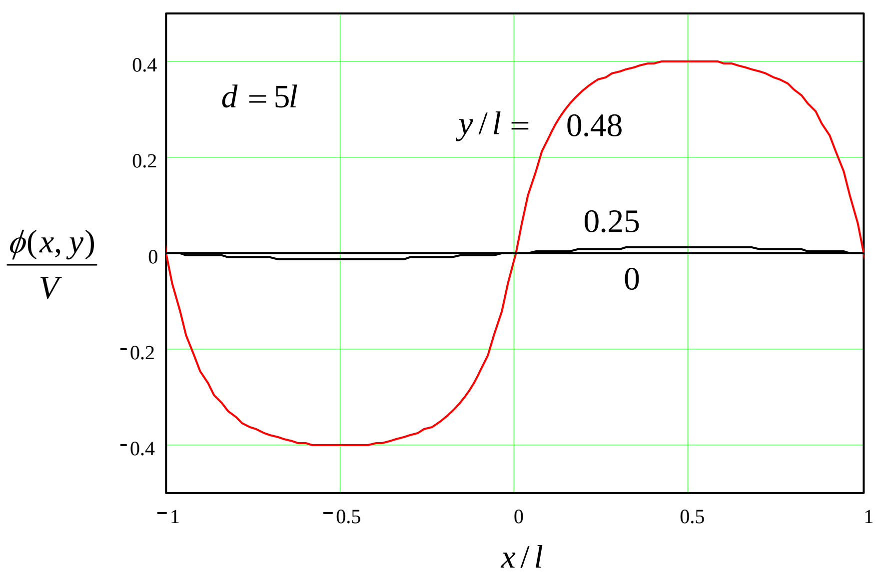
Applying the reciprocal Fourier transform formula (or just multiplying the both parts of the last equation by $\sin(\pi x n'/l)$ and integrating the result over x from 0 to l), we readily get

$$\phi_n = \begin{cases} \dfrac{2V}{\pi n}\left(\sinh\dfrac{\pi n d}{2l}\right)^{-1}, & \text{for } n \text{ odd}, \\ 0, & \text{for } n \text{ even}. \end{cases}$$

Two plots below show the potential distribution between the plates for two values of the ratio l/d. If $l << d$, the potential follows the square-wave profile, dictated by the electrodes, only very close (at distances $\Delta y \sim l$) to their surface, while it is close to the simple sine function, given by the first term on series (*), in most of the volume.[3]

[3] Though this was not a part of the posted problem, let me note that its solution in the outer regions ($|y| > d/2$) is similar, with $\sinh(\pi n y/l)$ replaced for $\text{sgn}(y)\exp\{-\pi n|y|/l\}$.


d = l
y / l = 0.48
0.25
0
φ(x, y) / V
x / l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0.25
0
φ(x, y) / V
x / l