

Problem 4.1 (to be graded of 15 points). Calculate the skin-effect contribution to the attenuation coefficient α defined by equation (7.196) of the lecture notes, for the basic (H_{10}) mode propagating in a waveguide with the rectangular cross-section (Fig. 7.21). Use the results to evaluate α for a 10 GHz wave propagating in the standard X-band waveguide WR-90 (with copper walls, $a = 23$ mm, $b = 10$ mm, and no dielectric filling) at room temperature. Compare the estimate with that for the standard coaxial cable, at the same frequency – see Sec. 7.9.

Solution: As discussed in class, in the H_{10} mode the electric field has just one Cartesian component, with complex amplitude

$$E_y(x) = i \frac{ka}{\pi} Z H_1 \sin \frac{\pi x}{a},$$

while the magnetic field has two components:

$$H_x(x) = -i \frac{k_z a}{\pi} H_1 \sin \frac{\pi x}{a}, \quad H_z(x) = H_1 \cos \frac{\pi x}{a}.$$

Of those two components, only H_x contributes to the longitudinal (z) component of the time-averaged Poynting vector

$$\overline{S}_z = \frac{E_x H_y^* - E_y H_x^*}{2} = \frac{k k_z a^2}{2\pi^2} Z |H_1|^2 \sin^2 \frac{\pi x}{a},$$

which gives the total power flow along the waveguide:

$$\overline{\mathcal{P}} = \int_0^a dx \int_0^b dy \overline{S}_z = \frac{k k_z a^3 b}{4\pi^2} Z |H_1|^2. \quad (*)$$

In order to find losses per unit length, we have to integrate losses per unit area, given by Eq. (7.206) of the lecture notes, over the cross-section's perimeter:

$$\begin{aligned} \frac{d\overline{\mathcal{P}}_{\text{loss}}}{dz} &= \frac{\mu_0 \omega \delta(\omega)}{4} \oint_C |H_0(x)|^2 dl = \frac{\mu_0 \omega \delta(\omega)}{4} \left[2 \int_0^a \left[|H_x(x)|^2 + |H_z(x)|^2 \right] dx + \int_0^b |H_z(0)|^2 dy + \int_0^b |H_z(b)|^2 dy \right] \\ &= \frac{\mu_0 \omega \delta(\omega)}{4} |H_1|^2 \left\{ 2 \int_0^a \left[\left(\frac{k_z a}{\pi} \right)^2 \sin^2 \frac{\pi x}{a} + \cos^2 \frac{\pi x}{a} \right] dx + 2b \right\} = \frac{\mu_0 \omega \delta(\omega)}{4} |H_1|^2 \left\{ \left[\left(\frac{k_z a}{\pi} \right)^2 + 1 \right] a + 2b \right\}. \end{aligned}$$

According to the waveguide's dispersion relation (see Eqs. (7.126) and (7.128) of the lecture notes),

$$k_z^2 + k_t^2 = k^2 = \omega^2 \varepsilon \mu, \quad (k_t)_{H_{10}} = \frac{\pi}{a},$$

the expression in the last square brackets is just $(ka/\pi)^2$, so that, using Eq. (*), for the attenuation constant we finally get

$$\alpha \equiv \frac{1}{\overline{\mathcal{P}}} \frac{d\overline{\mathcal{P}}_{\text{loss}}}{dz} = \pi^2 \frac{\mu\omega\delta(\omega)}{Zk k_z a^2 b} \left[\left(\frac{ka}{\pi} \right)^2 + \frac{2b}{a} \right] = \pi \frac{\delta(\omega)}{(k_z a / \pi) ab} \left[\left(\frac{ka}{\pi} \right)^2 + \frac{2b}{a} \right].$$

Notice that α scales approximately as $\delta(\omega)/A$, where $A \equiv ab$ is waveguide's cross-section area. More particularly, α diverges at $b \rightarrow 0$, because the transmitted power decreases while the loss remain constant (at fixed field amplitude). The dependence of the attenuation on frequency is non-monotonic: α diverges at $\omega \rightarrow \omega_c$, where $k_z \rightarrow 0$, but also grows slowly (as $\omega^{1/2}$) at $\omega \rightarrow \infty$, so that the lowest attenuation is reached $\sim 30\%$ above the threshold, where the higher modes (H_{11} , H_{20} , and E_{11}) still cannot propagate in the waveguide.

For a 10 GHz wave in the WR-90 waveguide we get $\lambda = \lambda_0 \approx 30$ mm, $ka/\pi = 2a/\lambda \approx 1.53$, $k_z a/\pi = [(ka/\pi)^2 - 1]^{1/2} \approx 1.16$, $\delta(\omega) \approx 6.5 \times 10^{-7}$ m, and $\alpha \approx 0.025$ m⁻¹ (i.e. ~ 0.1 db/m). Hence $\mathcal{L} \equiv 1/\alpha \approx 40$ m, so that the waveguide provides an attenuation well below that of the standard coaxial cable - mostly because of the waveguide's larger cross-section, rather than its different wave mode.

Problem 4.2 (15 points). For a rectangular resonator with dimensions $a \times b \times l$ ($b \leq a, l$), calculate the Q -factor in the basic (fundamental) oscillation mode, due to the skin-effect losses in metallic walls. Evaluate the factor for a $23 \times 23 \times 10$ mm³ resonator with copper walls, at room temperature.

Solution: Selecting the coordinates as shown in Fig. 29 of the lecture notes, the magnetic field distribution in the lowest (H_{110}) mode, discussed in Sec. 7.9, is as follows:

$$H_x = \frac{a}{l} H_l \sin \frac{\pi x}{a} \cos \frac{\pi z}{l}, \quad H_y = 0, \quad H_z = H_l \cos \frac{\pi x}{a} \sin \frac{\pi z}{l},$$

The energy of the EM field in the resonator may be calculated, for example, as the maximum energy of the magnetic field:

$$\begin{aligned} \overline{\mathcal{E}} &= \frac{\mu_0}{2} \int_0^a dx \int_0^b dy \int_0^l dz [H_x^2 + H_z^2] = \frac{\mu_0}{2} |H_l|^2 b \int_0^a dx \int_0^l dz \left[\frac{a^2}{l^2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi z}{l} + \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi z}{l} \right] \\ &= \frac{\mu_0}{8} |H_l|^2 abl \left(\frac{a^2}{l^2} + 1 \right). \end{aligned}$$

Now, the time-averaged power losses due to the skin-effect may be calculated virtually in the same way as in a waveguide (see the previous problem):

$$\begin{aligned} \overline{\mathcal{P}}_{\text{loss}} &= \frac{\mu_0 \omega \delta(\omega)}{4} \int_{\text{over all the walls}} |H_\tau(x, z)|^2 d^2 r \\ &= \frac{\mu_0 \omega \delta(\omega)}{4} \left\{ 2b \int_0^a |H_x(x, 0)|^2 dx + 2b \int_0^l |H_z(0, z)|^2 dz + 2 \int_0^a dx \int_0^l dz \left[|H_x(x, z)|^2 + |H_z(x, z)|^2 \right] \right\} \\ &= \frac{\mu_0 \omega \delta(\omega)}{4} |H_l|^2 l \left\{ b \left(\frac{a^3}{l^3} + 1 \right) + \frac{a}{2} \left(\frac{a^2}{l^2} + 1 \right) \right\}. \end{aligned}$$

As a result, for the Q -factor we get

$$Q \equiv \omega \frac{\bar{\mathcal{E}}}{\bar{\mathcal{P}}_{\text{loss}}} = \frac{1}{2\delta(\omega)} \left[ab \left(\frac{a^2}{l^2} + 1 \right) \right] \left[b \left(\frac{a^3}{l^3} + 1 \right) + \frac{a}{2} \left(\frac{a^2}{l^2} + 1 \right) \right]^{-1}.$$

As expected, Q scales as the ratio of some effective size of the resonator (of the order of wavelength λ), to skin depth δ . According to Eq. (7.189) of the lecture notes, the resonance frequency of the resonator specified in the assignment is $\omega \approx 6 \times 10^{10} \text{ s}^{-1}$ ($f \approx 10 \text{ GHz}$), so that the skin depth $\delta(\omega)$ for copper is close to $6.5 \times 10^{-7} \text{ m}$, and $Q \approx 8.2 \times 10^3$. For a microwave, room-temperature, metallic resonator, operating at their fundamental mode, this is almost as high quality as you can get. Note that the condition $Q \gg 1$ is well satisfied, so that our approximate method of calculation is indeed legitimate.

Problem 4.3 (20 points). In the electric dipole approximation, calculate the angular distribution and the total power of EM radiation by the classical (N. Bohr's) model of the hydrogen atom: an electron rotating, at a constant distance a , about a much heavier proton. Use the latter result to evaluate the classical lifetime of the atom, borrowing the initial value of a from quantum mechanics: $a(t=0) = a_B \approx 0.53 \times 10^{-10} \text{ m}$.

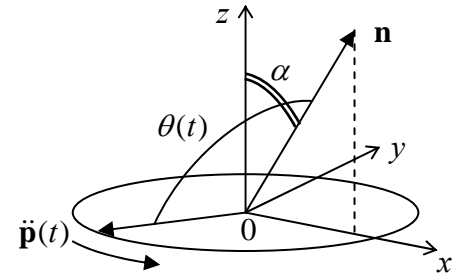
Solution: For the radial component of the instant Poynting vector, we can use Eq. (8.26) of the lecture notes:

$$S_r = \frac{Z}{(4\pi vr)^2} \ddot{\mathbf{p}}^2 \sin^2 \theta.$$

In our current problem, with the dipole moment $\mathbf{p} = q\mathbf{r}' = -e\mathbf{r}'$ (where \mathbf{r}' is the electron's position) rotates, angle θ is a function of time – see Fig. on the right. Its time average may be calculated in several equivalent ways, for example by returning to Eq. (8.24) to write

$$\ddot{\mathbf{p}}^2 \sin^2 \theta = |\mathbf{n} \times \ddot{\mathbf{p}}(t)|^2,$$

and calculating the vector product in some fixed coordinate system – for example, that shown in Fig. on the right, with axes x and y within the plane of the particle rotation, turned about axis z so that for the observation point, $y = 0$. In this system,



$$\ddot{\mathbf{p}}(t) = ea\omega^2 (\mathbf{n}_x \cos \Theta + \mathbf{n}_y \sin \Theta), \quad \mathbf{n} = \mathbf{n}_x \sin \alpha + \mathbf{n}_z \cos \alpha, \quad \text{with } \Theta \equiv \omega t + \text{const},$$

where ω is the angular velocity of electron rotation, so that the basic rule of vector multiplication (see, e.g., MA Eq. (7.3)) yields

$$\mathbf{n} \times \ddot{\mathbf{p}}(t) = ea\omega^2 (-\mathbf{n}_x \cos \alpha \sin \Theta + \mathbf{n}_y \cos \alpha \cos \Theta + \mathbf{n}_y \cos \alpha \cos \Theta + \mathbf{n}_z \sin \alpha \sin \Theta).$$

Averaging the square of this vector over the period of rotation, i.e. over interval $\Delta\Theta = 2\pi$, we readily get

$$\overline{(\ddot{\mathbf{p}} \sin \theta)^2} = e^2 a^2 \omega^4 \overline{(\cos^2 \alpha \sin^2 \Theta + \cos^2 \alpha \cos^2 \Theta + \sin^2 \alpha \sin^2 \Theta)} = e^2 a^2 \omega^4 \left(\cos^2 \alpha + \frac{1}{2} \sin^2 \alpha \right).$$

This formula shows that the angular distribution of the average radiated power is indeed different from that produced by an oscillating dipole of fixed orientation: the radiation is strongest along the direction normal to the charge rotation plane (in our notation, axis z), but is finite along any other direction. The total power of the radiation (into free space)

$$\overline{\mathcal{P}} = \int_{4\pi} \overline{S}_r r^2 d\Omega = Z_0 \left(\frac{\omega^2 ea}{4\pi c} \right)^2 2\pi \int_0^\pi \left(\cos^2 \alpha + \frac{1}{2} \sin^2 \alpha \right) \sin \alpha d\alpha = Z_0 \left(\frac{\omega^2 ea}{4\pi c} \right)^2 2\pi \frac{4}{3} = \frac{Z_0 \omega^4 e^2 a^2}{6\pi c^2}.$$

Comparing this result with Eq. (8.29) for a fixed-direction dipole, we see that the radiation power by a rotating charge equals a sum of those from its two oscillating components $p_x(t)$ and $p_y(t)$, calculated independently. The reason of this independence (which is not immediately apparent) is that the polarizations of the partial waves radiated by the 1D component dipoles are mutually perpendicular.

Proceeding to the calculation of the classical lifetime of the Bohr atom, from the elementary classical mechanics we know that at the circular motion of a nonrelativistic particle in the Coulomb attractive field (with potential energy $U(r) = -e^2/4\pi\epsilon_0 r$), kinetic energy $T \equiv mv^2/2$ equals $-U(a)/2$, so that the full energy $\mathcal{E} \equiv T + U$ equals $U(a)/2$. In our case, this means

$$\mathcal{E} = -\frac{e^2}{8\pi\epsilon_0 a}, \quad T = \frac{e^2}{8\pi\epsilon_0 a}, \quad \text{so that } \omega^2 = \frac{v^2}{a^2} = \frac{2T/m}{a^2} = \frac{e^2}{4\pi\epsilon_0 m a^3}.$$

Weak radiation with of power $\overline{\mathcal{P}} \ll \omega E$ would cause a relatively slow energy reduction: $dE/dt = -\overline{\mathcal{P}}$, in our particular case giving

$$-\frac{e^2}{8\pi\epsilon_0} \frac{d}{dt} \left(\frac{1}{a} \right) = -Z_0 \frac{\omega^4 e^2 a^2}{6\pi c^2} = -\frac{Z_0 e^6}{6\pi c^2 a^4 (4\pi\epsilon_0)^2 m^2} < 0.$$

This equation, describing a monotonic *increase* of $(1/a)$, i.e., the electron's fall on the proton nucleus, may be rewritten as

$$\tau \frac{d\xi}{dt} = -\frac{1}{3\xi^2}, \quad \tau \equiv \frac{4\pi^2 m^2 a^3(0)}{e^4 Z_0 \mu_0}$$

where $\xi \equiv a(t)/a(0)$. It may be readily integrated:

$$\xi(t) = \left(1 - \frac{t}{\tau} \right)^{1/3},$$

showing that $a(t)$ vanishes (i.e. the electron drops on the proton) at $t = \tau$.

For an electron ($e \approx -1.6 \times 10^{-19}$ C, $m = m_e \approx 0.91 \times 10^{-30}$ kg), initially rotating at the Bohr radius, $a(0) = a_B \approx 0.53 \times 10^{-10}$ m, we get $\tau \approx 1.57 \times 10^{-11}$ s. Such the atom's collapse is very fast on the human scale of events (let us thank quantum mechanics for preventing this disaster!), but since for our parameters $\omega \sim 4 \times 10^{16}$ s⁻¹, the same process is slow on the rotation period scale: $\omega\tau \sim 10^4 \gg 1$. This strong relation justifies our approach which neglects the loss of energy during one rotation period at the average radiated power calculation stage. Another necessary numerical check is that the radiation wavelength scale, $\lambda = 2\pi c/\omega \sim 5 \times 10^{-8}$ m, is much larger than $a \sim 10^{-10}$ m, so that the radiating charge can be indeed treated as a point electric dipole, as it was.