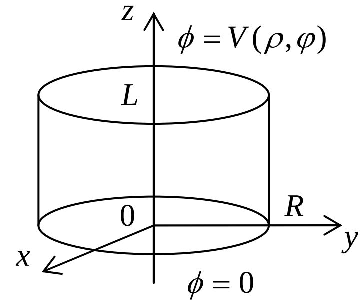
PHY 505 **Classical Electrodynamics I** Fall 2010

Homework 4 with solutions



Problem 4.1 (to be graded of 20 points). Complete the cylinder problem started in class (see Fig. on the right), for the cases when voltage on the top lid equals:

(i) $V = V_0 J_1(\xi_{11}\rho/R)\sin\varphi$, where $\xi_{11} \approx 3.832$ is the first root of function $J_1(x)$, and

(ii) $V = V_0 =$ const.

For both cases, calculate the electric field in the centers of the lower and upper lids.

Hint: For assignment (ii), an answer including series and/or integrals is satisfactory.

Solution: In class, we have found the general solution to this problem:

$$\phi(\rho,\varphi,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\frac{\xi_{nm}\rho}{R})\left(c_{nm}\cos n\varphi + s_{nm}\sin n\varphi\right)\sinh\frac{\xi_{nm}z}{R}, \qquad (*)$$

where ξ_{nm} is the m-th root of the Bessel function $J_n(x)$. This solution already satisfies the boundary conditions on the sidewall and the bottom lid of the cylinder; hence coefficients c_n and s_n have to be found from the boundary condition on the top lid:

$$\phi(\rho,\varphi,L) = V(\rho,\varphi). \qquad (**)$$

In assignment (i), this function has only one azimuthal harmonic proportional to sinφ, and only one radial harmonic, proportional to $J_1(\xi_{11}\rho/R)$, so that only one of the coefficients c_{nm}, s_{nm} is not vanishing:

$$s_{11}\sinh\frac{\xi_{11}L}{R} = V_0,$$

and Eq. (*) is reduced to a very simple analytical solution

$$\phi = V_0 J_1(\frac{\xi_{11}\rho}{R})\frac{\sinh\frac{\xi_{11}z}{R}}{\sinh\frac{\xi_{11}L}{R}}\sin\varphi.$$

Since $J_1(0) = 0$, the potential equals zero along all the symmetry axis of the system, so that in this case there is no normal electric field $E_n = -\partial\phi/\partial z$ in the center of either lid. However, since the potential changes within the top lid plane,

$$V = V_0 J_1\left(\frac{\xi_{11}\rho}{R}\right)\sin\varphi_{\rho\to 0} \to V_0\frac{\xi_{11}\rho}{2R}\sin\varphi = V_0\frac{\xi_{11}}{2R}y,$$

is does have a horizontal component directed along axis y:

$$E_y\Big|_{\rho=0} = -V_0 \frac{\xi_{11}}{2R} \approx -1.916 \frac{V_0}{R}.$$

In assignment (ii), voltage on the top lid does not depend of angle φ, and hence only one angular function (with $n = 0$) fits this boundary condition, and Eq. (**) is reduced to an axially-symmetric form,

$$V_0 = \sum_{m=1}^{\infty} c_{0m} J_0\left(\frac{\xi_{0m}\rho}{R}\right) \sinh \frac{\xi_{0m} L}{R}.$$

Multiplying both parts of this equation by $\rho J_0(x_{0m}\rho/R)$, integrating the result over ρ from 0 to R, and using Eq. (2.141) of the lecture notes, we get

$$c_{0m} = \frac{2V_0}{R^2 J_1^2(\xi_{0m}) \sinh \dfrac{\xi_{0m} L}{R}} \int_0^R J_0\left(\xi_{0m} \frac{\rho}{R}\right) \rho\, d\rho.$$

(The last integral may be worked out analytically, giving a more explicit result:

$$c_{0m} = \frac{2V_0}{\xi_{0m} J_1(\xi_{0m}) \sinh \dfrac{\xi_{0m} L}{R}},$$

but I could not expect my PHY 505 students to accomplish such an astonishing feat :-)

According to Eq. (*), and due to the fact that $J_n(0) = 0$ unless $n = 0$ (see, e.g., Fig. 2.16), the potential distribution along the symmetry axis is contributed only by terms with $n = 0$ alone:

$$\phi(z) = \sum_{m=1}^{\infty} c_{0m} \sinh \frac{\xi_{0m} z}{R},$$

so that the (vertical) field in the center of the bottom lid ($z = 0$) is

$$E_n\Big|_{\text{bottom}} = E_z\Big|_{z=0} = -\frac{d\phi(z)}{dz}\Big|_{z=0} = -\sum_{m=1}^{\infty} c_{0m} \frac{\xi_{0m}}{R}, \qquad (***)$$

while the field at the top lid ($z = L$) is

$$E_n\Big|_{\text{top}} = -E_z\Big|_{z=L} = \frac{d\phi(z)}{dz}\Big|_{z=L} = \sum_{m=1}^{\infty} c_{0m} \frac{\xi_{0m}}{R} \cosh \frac{\xi_{0m} L}{R}.$$

For any $L > 0$, each term of the last series is larger that the corresponding term of Eq. (***), so that the field magnitude at the bottom lid is always lower, due to the screening effects of the side wall of the cylinder, though at $L/R \to 0$, $\cosh(\xi_{0m} L/R) \to 1$ for all major terms in the series, and this difference is negligible.

Problem 4.2 (20 points). Use the variable separation method to find the potential distribution outside and inside a thin spherical shell of radius R, with fixed potential $\phi(R,\theta,\varphi) = V_0 \sin\theta \cos\varphi$.

Solution: From Eq. (2.182) of the lecture notes, we see that the surface potential is proportional to product $\mathscr{P}^1_1(\cos\theta)\mathcal{F}_1(\varphi)$, i.e. to a single angular function with $l = 1$ and $n = 1$. Inside the sphere, the particular solution of the Laplace equation, proportional to this function, can use only radial functions $a_l r^l$ which do not diverge at $r \to 0$, while for the outer problem we may only use functions b_l/r^{l+1} which do not diverge at $r \to \infty$. As a result, the general solution to the Laplace equation in spherical coordinates is reduced, for our case, to

$$\phi(r,\theta,\varphi) = \begin{cases} a_1 r \sin\theta\cos\varphi, & \text{for } r \leq R, \\ \dfrac{b_1}{r^2}\sin\theta\cos\varphi, & \text{for } R \leq r. \end{cases}$$

Finding constants a_1 and b_1 from the boundary conditions on the shell ($r = R$), we get finally

$$\phi(r,\theta,\varphi) = V_0 \begin{cases} (r/R)\sin\theta\cos\varphi, & \text{for } r \leq R, \\ (R/r)^2\sin\theta\cos\varphi, & \text{for } R \leq r. \end{cases}$$

Note that the first of these results may be simply expressed in Cartesian coordinates,

$$\phi = \frac{V_0}{R} x, \quad \text{for } r \leq R,$$

so that the field inside the sphere is uniform: $\mathbf{E} = -(V_0/R)\mathbf{n}_x$.