PHY 506 **Classical Electrodynamics II** Spring 2011

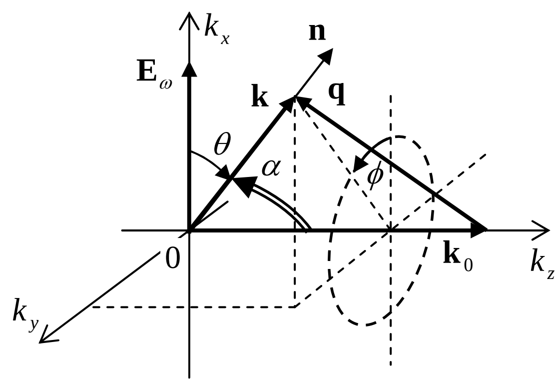
Homework 5 with solutions

Problem 5.1 (25 points). Use the Born approximation to calculate the differential cross-section of plane wave scattering by a dielectric sphere with $\varepsilon \approx \varepsilon_0$, of an arbitrary radius R. In the limits $kR << 1$ and $kR >> 1$ (where k is the wave vector), analyze the angular dependence of the differential cross-section, and calculate the full cross-section.

Solution: According to Eqs. (8.62)-(8.63) of the lecture notes, the differential cross-section may be calculated as

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi)^2}(\varepsilon_r - 1)^2 \left|I(\mathbf{q})\right|^2 \sin^2\theta, \quad I(\mathbf{q}) = \int_V \exp\{-i\mathbf{q}\cdot\mathbf{r}\} d^3r ,$$

where the phase integral $I(\mathbf{q})$ is taken over the scatterer's volume, and vector $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$ is the wave vector change due to scattering. According to Fig. 8.5 (reproduced on the right), magnitude of $\mathbf{q}$ is related to the scattering angle α (i.e. the angle between vectors $\mathbf{k}$ and $\mathbf{k}_0$) as $q = 2k\sin(\alpha/2)$. As this figure shows, angle θ (i.e. the angle between vectors $\mathbf{E}_\omega$ and $\mathbf{k}$) is related to α as follows:



$$\sin^2\theta = \frac{k_y^2 + k_z^2}{k^2} = \sin^2\alpha\sin^2\phi + \cos^2\alpha ,$$

where ϕ is the azimuthal angle of scattering (defined as shown in the figure).

For a sphere, it is natural to work out the phase integral by taking the direction of vector $\mathbf{q}$ for the polar axis, so that $\mathbf{q}\cdot\mathbf{r} = qr\cos\Theta$ (notice that $\Theta \neq \theta$!), and

$$I(\mathbf{q}) = 2\pi\int_0^R r^2 dr \int_0^\pi \exp\{-iqr\cos\Theta\}\sin\Theta d\Theta = -2\pi\int_0^R r^2 dr\int_0^\pi \exp\{-iqr\cos\Theta\} d(\cos\Theta)$$

$$= 2\pi\int_0^R r^2 dr \frac{1}{-iqr}\left(e^{-iqr} - e^{+iqr}\right) = \frac{4\pi}{q}\int_0^R r dr \sin qr = \frac{4\pi}{q^3}\left[\sin qR - (qR)\cos qR\right].$$

As a result, the differential cross-section is

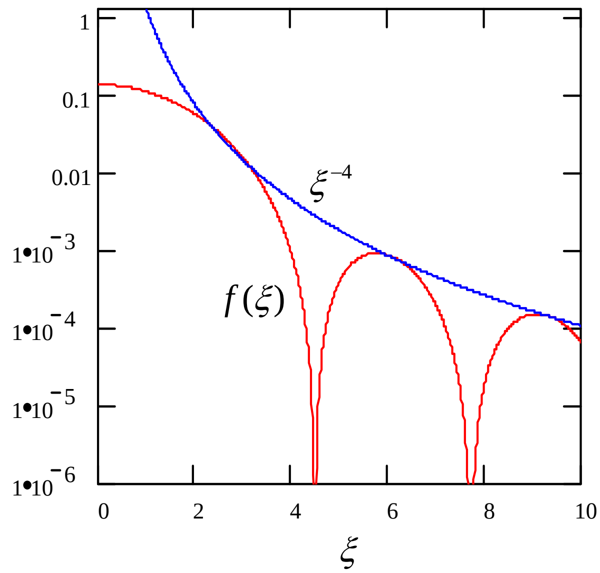
$$\frac{d\sigma}{d\Omega} = k^4 R^6 (\varepsilon_r - 1)^2 \sin^2\theta\, f(qR), \qquad f(\xi) \equiv \frac{\left[\sin\xi - \xi\cos\xi\right]^2}{\xi^6}. \qquad (*)$$

For a relatively large sphere, $kR >> 1$, function $f(qR)$ oscillates already at relatively small values of $q \sim 1/R$, and hence α, at which $\theta \approx \pi/2$, so that the cross-section follows the function – see the red line in Fig. below. These oscillations describe the round-ring diffraction pattern, with positions of minima rapidly approaching the asymptotic values

$$q_n = (n + \frac{1}{2})\frac{\pi}{R}, \quad n = 1,2,\ldots,$$

i.e. scattering angles

$$\alpha_n \approx \frac{q_n}{k} = (n + \frac{1}{2})\frac{\pi}{kR} << 1.$$



(Notice that the diffraction pattern is absent at "backscattering", i.e. at $\alpha \approx \pi$, because vector q in that range is large (approaching $2k$). As qR is increased, the oscillations envelope is dropping very fast: as $(qR)^{-4}$ – see the blue line in Fig. above. As a result, the total cross-section integral over the solid angle converges already at small scattering angles $\alpha << 1$ where we may accept $\sin^2\theta = 1$, $q = k\alpha$, and $\sin\alpha = \alpha$, so that

$$\sigma \approx \int_{4\pi} \frac{d\sigma}{d\Omega}\Big|_{\alpha<<1} d\Omega = 2\pi k^4 R^6 (\varepsilon_r - 1)^2 \int_0^{\alpha_m} f(kR\alpha)\alpha d\alpha = 2\pi k^2 R^4 (\varepsilon_r - 1)^2 \int_0^{\infty} f(x) x dx = \frac{1}{2}\sigma_0 (kR)^2 (\varepsilon_r - 1)^2,$$

where $\sigma_0 \equiv \pi R^2$ is the geometrical cross-section of the sphere. Since $kR >> 1$, it is tempting to think that σ described by this formula may be much larger than σ_0. However, according to Eq. (8.48), the Born approximation is only valid if $\sigma << \sigma_0$, i.e. if $(\varepsilon_r - 1) << 1/kR$.

In the opposite limit, $kR << 1$, we can use Eq. (*) with $qR << 1$ and $f(qR) \approx f(0) = 1/9$ for all possible directions of vector **k** ($0 \le q \le 2k$), so that

$$\frac{d\sigma}{d\Omega} = \frac{1}{9} k^4 R^6 (\varepsilon_r - 1)^2 \sin^2\theta.$$

This result coincides with the general Eq. (8.53) of the lecture notes (with $V = (4\pi/3)R^3$) and shows that in this limit, there is no diffraction pattern to speak about. As we have already seen at the derivation of Eq. (8.27), the solid-angle integral of $\sin^2\theta$ equals $8\pi/3$, so that the full cross-section in this limit,

$$\sigma = \frac{8\pi}{27} k^4 R^6 (\varepsilon_r - 1)^2 = \frac{8}{27}\sigma_0 (kR)^4 (\varepsilon_r - 1)^2.$$