

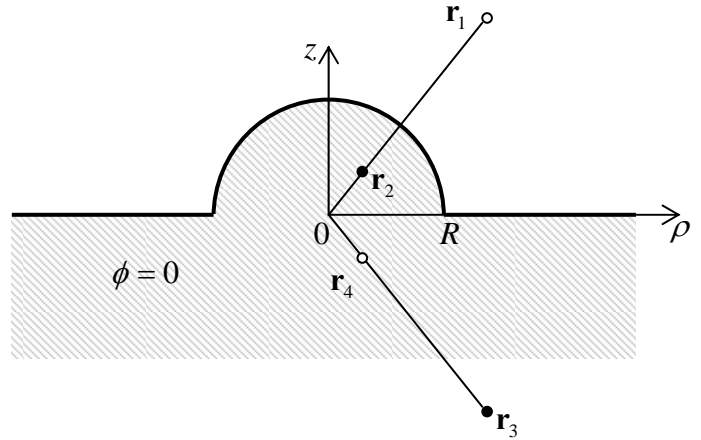
**Problem 5.1** (to be graded of 10 points). Use the method of images to find the Green's function of the system shown in Fig. on the right. (The bulge on the conducting plane has the shape of a semi-sphere of radius  $R$ .)

*Solution:* Let a (real) point charge  $q_1$  be at point  $\mathbf{r}_1$ , with  $r_1 \geq R$ . Then all the boundary conditions may be satisfied using three charge images (see Fig.), with

$$q_2 = -q_1 \frac{R}{r_1}, \quad \mathbf{r}_2 = \left(\frac{R}{r_1}\right)^2 \mathbf{r}_1,$$

$$q_3 = -q_1, \quad \boldsymbol{\rho}_3 = \boldsymbol{\rho}_1, \quad z_3 = -z_1,$$

$$q_4 = -q_2 = q_1 \frac{R}{r_1}, \quad \mathbf{r}_4 = \left(\frac{R}{r_3}\right)^2 \mathbf{r}_3,$$



where  $\boldsymbol{\rho}_j$  is the horizontal component of radius-vector  $\mathbf{r}_j$ , and  $z_j$  is its vertical component. As a result, the Green's function may be presented simply as

$$G(\mathbf{r}, \mathbf{r}') = \sum_{j=1}^4 \frac{q_j}{q_1} \frac{1}{|\mathbf{r} - \mathbf{r}_j|}, \quad \mathbf{r}_1 = \mathbf{r}'.$$

However, even in the most suitable, cylindrical coordinates, this compact expression becomes rather bulky, because each denominator looks like

$$|\mathbf{r} - \mathbf{r}_j| = \left[ \rho^2 + \rho_j^2 + 2\rho\rho_j \cos(\varphi - \varphi_j) + (z - z_j)^2 \right]^{1/2},$$

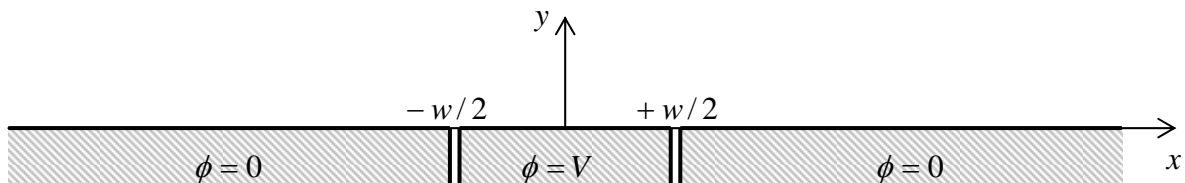
while

$$r_1 = (\rho_1^2 + z_1^2)^{1/2}.$$

**Problem 5.2** (15 points). Find the 2D Green's function in:

- (i) unlimited free space, and
- (ii) free space above a conducting plane.

Use the latter result to calculate the distribution of the electric potential created by the 2D system shown in Fig. below. (The gaps between the conducting fragments are negligibly small.)



*Solutions:*

(i) In analogy with Eq. (2.208) of the lecture notes, the 2D Green's function may be defined as the solution of 2D Laplace equation<sup>1</sup>

$$\nabla_2^2 G(\mathbf{\rho}, \mathbf{\rho}') = -4\pi\delta_2(\mathbf{\rho} - \mathbf{\rho}').$$

This equation shows that  $G$  may be interpreted as the electrostatic potential created by a thin, straight filament, with uniform charge density  $\lambda = 4\pi\epsilon_0$ , at distance  $|\mathbf{\rho} - \mathbf{\rho}'|$  from it. We already know the solution to that problem:

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \ln \rho + \text{const},$$

so that we get

$$G(\mathbf{\rho}, \mathbf{\rho}') = -2 \ln |\mathbf{\rho} - \mathbf{\rho}'| + \text{const} = -\ln |\mathbf{\rho} - \mathbf{\rho}'|^2 + \text{const}.$$

(ii) For a charged line parallel to a grounded conducting plane  $y = 0$ , crossing the plane of drawing in point  $\mathbf{\rho} = \{x, y\}$ , both the Poisson equation and boundary condition,  $\phi(x, 0) = 0$ , may be satisfied by the introduction of a image line with charge density  $-\lambda$ , crossing the plane of drawing in point  $\mathbf{\rho}'' = \{x', -y'\}$ . Hence the Green's function of the system is

$$G(\mathbf{\rho}, \mathbf{\rho}') = -\ln |\mathbf{\rho} - \mathbf{\rho}'|^2 + \ln |\mathbf{\rho} - \mathbf{\rho}''|^2 = -\ln[(x - x')^2 + (y - y')^2] + \ln[(x - x')^2 + (y + y')^2]. \quad (*)$$

For the particular problem shown in Fig. above, all we need is the normal component of the gradient of the Green's function at the surface. A straightforward differentiation of Eq. (\*) yields

$$\left. \frac{\partial G}{\partial n'} \right|_A = \left. \frac{\partial G}{\partial y'} \right|_{y'=0} = \frac{4y}{(x - x')^2 + y^2}.$$

Plugging this expression into the 2D version of the main formula of the Green's function theory (with zero free charge), we get

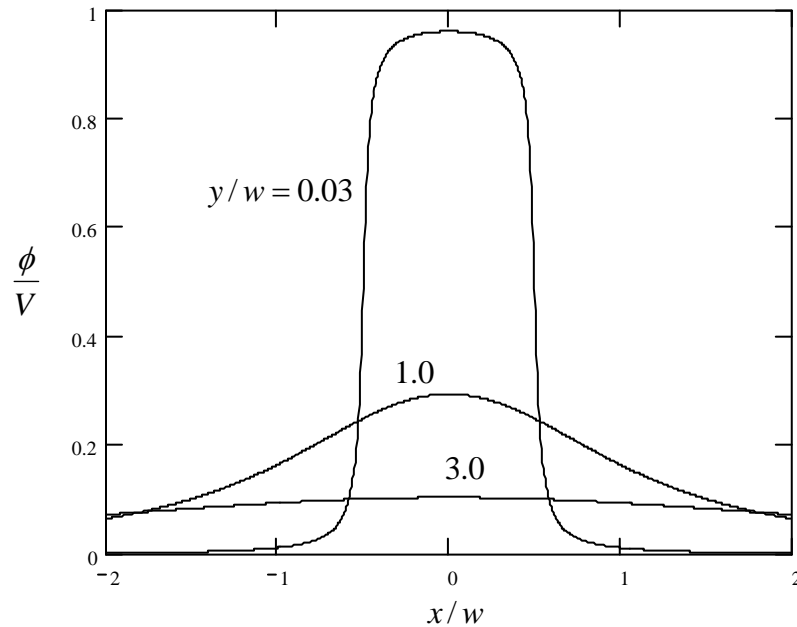
$$\phi(\mathbf{\rho}) = \frac{1}{4\pi} \sum_k \phi_k \oint_{L_k} \frac{\partial G(\mathbf{\rho}, \mathbf{\rho}')}{\partial n'} dr' = \frac{V}{4\pi} \int_{-w/2}^{+w/2} \frac{4y}{(x - x')^2 + y^2} dx'.$$

This integral may be readily taken using the substitution  $\xi \equiv (x - x')/y$ , giving

$$\phi = -\frac{V}{\pi} \int_{(x+W/2)/y}^{(x-W/2)/y} \frac{d\xi}{\xi^2 + 1} = \frac{V}{\pi} \left[ \arctan \frac{x + w/2}{y} - \arctan \frac{x - w/2}{y} \right].$$

As the numerical plot of this result (see Fig. below) shows, it result describes a "bump"-shaped distribution of the potential along axis  $x$ . Close to the conducting plane, the bump is sharp, and its height approaches  $V$ , but at large distances from the surface the bump is lower and more smooth, with width  $\Delta x \sim y$ .

<sup>1</sup> Actually, the coefficient in the right-hand part of this equation is the matter of convention.



**Problem 5.3** (15 points). Solve the same 2D boundary problem which has been discussed in class (Fig. 2.32 of the lecture notes) using:

- (i) the finite difference method, with a finer square mesh,  $h = a/3$ , and
- (ii) the variable separation method.

Compare the results (in the mesh points only) and comment.

*Solutions:*

(i) Due to the problem symmetry, there are only two types of internal nodes in this mesh: A and B (see Fig. on the right), so we need only two finite difference equations, each describing a 5-point “cross” with the center in each of these points:

- point A:

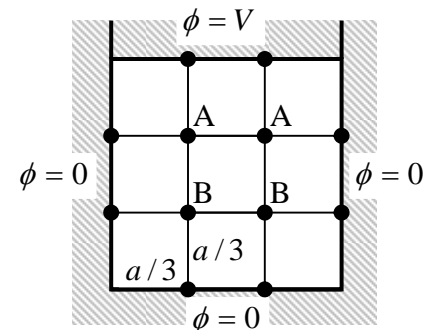
$$0 + \phi_A + V + \phi_B - 4\phi_A = 0,$$

- point B:

$$0 + \phi_B + \phi_A + 0 - 4\phi_B = 0.$$

Solving this simple system of equations, we get

$$\phi_A = \frac{3}{8}V = 0.375V, \quad \phi_B = \frac{1}{8} = 0.125V.$$



(ii) The solution of this 2D boundary Laplace problem using the variable separation is

$$\phi(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{\pi n x}{a} \sinh \frac{\pi n y}{a},$$

where coefficients  $c_n$  have to be found from the boundary condition on the top lid ( $y = a$ ):

$$V = \sum_{n=0}^{\infty} c_n \sin \frac{\pi n x}{a} \sinh \pi n.$$

Performing the reciprocal Fourier transform (i.e., multiplying both parts of this equation by  $\sin(\pi n' x/a)$ , and integrating over  $x$  from 0 to  $a$ ), we get

$$c_n = \frac{1}{(a/2) \sinh \pi n} V \int_0^a \sin \frac{\pi n x}{a} dx = \begin{cases} 4V / \pi n \sinh \pi n, & \text{for } n = 2j + 1, \\ 0, & \text{for } n = 2j, \end{cases} \quad j = 0, 1, 2, \dots$$

For the potentials in the  $a/3$  mesh nodes, these equations yield

$$\phi_A = \frac{4}{\pi} V \sum_{j=0}^{\infty} \frac{\sinh[2\pi(2j+1)/3]}{(2j+1) \sinh[\pi(2j+1)]} \approx 0.381 V,$$

$$\phi_B = \frac{4}{\pi} V \sum_{j=0}^{\infty} \frac{\sinh[\pi(2j+1)/3]}{(2j+1) \sinh[\pi(2j+1)]} \approx 0.119 V.$$

Notice that though the sums are formally infinite, virtually all of their values (but a few percent) are provided by the first terms, because the factor  $(2j+1)$  in the denominator and the exponential character of the  $\sinh$  function ensure a fast convergence of the series.

Comparing the results provided by the two methods, we see that the finite difference method's error, even with this rather crude mesh, is of the order of just a few percent.