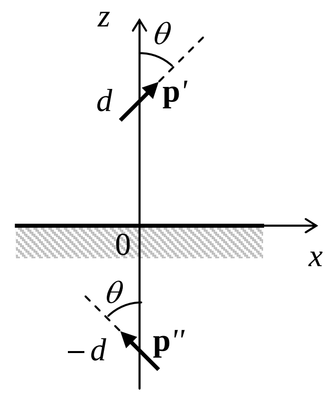
PHY 505 **Classical Electrodynamics I** Fall 2010

Homework 6 with solutions

Problem 6.1 (to be graded of 10 points). An electric dipole is located above an infinite conducting plane. Calculate:

(i) the distribution of the induced charge in the conductor,
(ii) the force and the torque acting on the dipole, and
(iii) the dipole-to-plane interaction energy.



Solutions:

(i) The problem may be solved by the introduction of a dipole image, at the same distance d below the plane, and with the same dipole moment magnitude p as the original dipole, but reflected in the vertical plane perpendicular to that containing the dipole moment vector (see Fig. on the right).[1] Let us prove that. The net field of these two dipoles evidently satisfies the Poisson equation in the upper half-space, so that the only thing we have to prove is that it also satisfies the boundary condition ($\phi = 0$) on the plane surface. Let us use Eq. (3.7) of the lecture notes for of a system of several dipoles – in our case, of two dipoles (let us call them **p**' and **p**"), with Cartesian components

$$p'_x = -p''_x = p\sin\theta, \quad p'_y = p''_y = 0, \quad p'_z = p''_z = p\cos\theta,$$

located at points

$$x' = x'' = 0, \quad y' = y'' = 0, \quad z' = -z'' = d.$$

(Here x is the coordinate within the vertical plane which contains vectors **p**' and **p**", i.e. in the plane of our drawing, while axis y is perpendicular to the plane.) In these coordinates, Eq. (3.7) yields

$$\phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}'}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{(\mathbf{r}-\mathbf{r}'')\cdot\mathbf{p}''}{|\mathbf{r}-\mathbf{r}''|^3}\right] = \frac{p}{4\pi\varepsilon_0}\left[\frac{(z-d)\cos\theta + x\sin\theta}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} + \frac{(z+d)\cos\theta - x\sin\theta}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right].$$

This equation shows that potential vanishes at an arbitrary point of the surface ($z = 0$), thus proving our guess.

Now the induced charge may be calculated as

$$\sigma = -\varepsilon_0 \frac{\partial\phi}{\partial z}\Big|_{z=0},$$

giving

[1] The simplest way to understand this fact is to present the dipole in the form of two point charges (+q) and (-q), slightly displaced along the direction of the dipole moment vector, and to construct the dipole image from the mirror images of these point charges in the conducting plane. However, this approach, based on a particular implementation of a dipole, and can only be used for a *guess*, not as a *proof*.

$$\sigma = \frac{p}{2\pi}\frac{(2d^2 - x^2 - y^2)\cos\theta - 3dx\sin\theta}{(x^2 + y^2 + d^2)^{5/2}}.$$

(iii) Now, we can use Eqs. (3.16) to calculate the potential energy of interaction force between the real and imaginary dipoles (e.g., between the dipole and the plane). However, we should not forget that the image has been created by the dipole itself. Hence, following the reasoning of Sec. 1.3, we should multiply the right-hand part of Eq. (3.16) by ½:

$$U_{\text{int}} = -\frac{1}{2}\mathbf{p}'\cdot\mathbf{E}''(\mathbf{r}'), \qquad (*)$$

where **E**''(**r**') is the electric field of dipole **p**'' at point **r**'.[2] According to Eq. (3.13) of the lecture notes, in our coordinates

$$E_x'' = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{(2d)^3}, \quad E_y'' = 0, \quad E_z'' = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{(2d)^3}.$$

With that, Eq. (*) yields[3]

$$U_{\text{int}} = -\frac{1}{8\pi\varepsilon_0}\frac{p^2}{(2d)^3}\left(1+\cos^2\theta\right). \qquad (**)$$

Notice that for any angle θ, the interaction energy is negative, i.e. the dipole is always attracted to a conductor. (Try to give an interpretation of this fact, presenting the dipole as a couple of point charges.)

(ii) Now we can use Eq. (**) to calculate the force and torque. As should be clear from the symmetry of this expression (namely, its independence on the horizontal position of the dipole), the force has only one nonvanishing component,

$$F_z = -\frac{\partial U_{\text{int}}}{\partial d} = -\frac{1}{4\pi\varepsilon_0}\frac{3p^2}{16d^4}\left(1+\cos^2\theta\right).$$

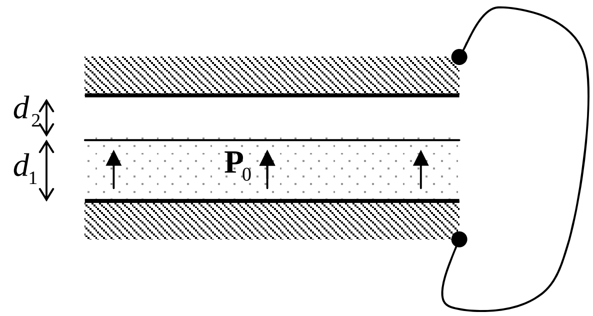
The torque vector also has only one component, but perpendicular to the plane of drawing:

$$\tau_y = -\frac{\partial U_{\text{int}}}{\partial\theta} = -\frac{1}{4\pi\varepsilon_0}\frac{p^2}{16d^3}\sin 2\theta.$$

(The same result may be obtained from Eq. (3.17) of the lecture notes) It is interesting that the torque disappears both at $\theta = 0$, π, and $\theta = \pm\pi/2$, i.e. in the positions in which the dipole moment is aligned along the field created by its image. Of these configurations, the former two, $\theta = 0$ (both dipoles up), and $\theta = \pi$ (both dipoles down), are stable with respect to dipole rotation, because they correspond to the interaction energy minima. This bistable system may serve as a toy model of a ferroelectric memory cell.

[2] Due to reciprocity property of electrostatics, -½**p**''·**E**'(**r**'') would give the same result.
[3] The same result follows from the equation given in the footnote on p. 4 of the lecture notes, after it is multiplied by the same factor ½.

Problem 6.2 (10 points). A plane capacitor, with zero voltage between its conducting plates (as may be fixed, e.g., with an external wire – see Fig. on the right), is partly filled with a material with spontaneous, constant polarization $\mathbf{P}_0$. Find the distributions of the electric field, electric displacement, and the surface charge density of each conducting plate.



Solution: In this symmetric configuration, vectors **E**, **P**, and **D** are all evidently vertical, so that we can limit our analysis to their *z*-components *E*, *P*, and *D*. Also, it is evident that in each of the layers each of the components is constant. (This follows from the 1D Laplace equation valid for the electric potential in each layer.) Next, since there is no free surface charge on the interface between two layers, the electric displacement in both layers should be the same: $D_1 = D_2 = D$. In the lower layer (with fixed polarization), definition (3.28) of vector **D** yields

$$D = \varepsilon_0 E_1 + P_0,$$

while in the upper (vacuum) layer,

$$D = \varepsilon_0 E_2 .$$

Also, since the voltage between the plates is zero, we should also require that the integral of E , taken between the plates, vanishes. This gives us one more equation:

$$E_1 d_1 + E_2 d_2 = 0.$$

Solving the system of these 3 equations for 3 unknown variables (D, E_1, and E_2), we get.

$$E_1 = -\frac{P_0}{\varepsilon_0}\frac{d_2}{d_1+d_2}, \quad E_2 = \frac{P_0}{\varepsilon_0}\frac{d_1}{d_1+d_2}, \quad D_1 = D_2 = D = P_0\frac{d_1}{d_1+d_2}.$$

Now the densities of charge on electrode surfaces may be readily found from Eq. (3.35):

$$\sigma_1 = -\sigma_2 = D = P_0\frac{d_1}{d_1+d_2}.$$

Notice that due to the spontaneous polarization of the lower layer material,[4] the capacitor plates are charged even in the absence of voltage between them. Also, the charge is a function of the second electrode position (d_2). This effect is used in simple microphones.

Problem 6.3 (10 points). A point charge q is located at distance $r >> R$ from the center of a uniform sphere of radius R, made of a linear dielectric. In the first nonvanishing approximation in small parameter R/r, calculate:

(i) the interaction force, and
(ii) the energy of interaction between the sphere and the charge.

[4] In electrical engineering, such materials (typically synthetic polymers) are frequently called *electrets*.

Solution: The point charge field $E = (1/4\pi\varepsilon_0)q/r^2$ is nearly uniform on the scale $\sim R \ll r$. From the problem solved in class (see Sec. 3.4 of the lecture notes), we know that such uniform field induces in a dielectric sphere a dipole moment of magnitude $p = 4\pi\varepsilon_0 ER^3(\varepsilon_r - 1)/(\varepsilon_r + 2)$, directed along the initial electric field – see the first of Eqs. (3.46). Hence we can use the general formula for the radial component of the dipole field, $E_r = 2p\cos\theta/4\pi\varepsilon_0 r^3$, with $\theta = 0$, to calculate the interaction (attraction) force magnitude:

$$F = qE_r = q\frac{2}{4\pi\varepsilon_0 r^3}p = q\frac{2}{4\pi\varepsilon_0 r^3}4\pi\varepsilon_0 R^3\frac{\varepsilon_r - 1}{\varepsilon_r + 2}E = q\frac{2}{r^3}R^3\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0}2\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\frac{q^2R^3}{r^5}.$$

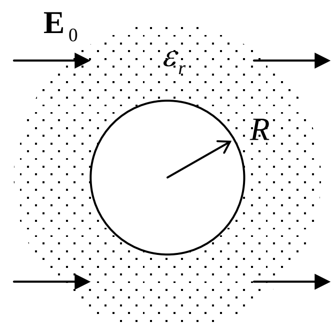
Integrating the result from ∞ to r, we get the potential energy of the charge-sphere interaction:

$$U = -\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\frac{q^2R^3}{r^4}.$$

Another way to obtain the last result is to use Eq. (3.16) for the interaction energy of a dipole with an external electric field, but with the right-hand part multiplied by factor ½, in order to account for the fact that in our case the dipole moment is induced by the external field itself. In this approach, the interaction force magnitude may be obtained at the second step, by calculating the (minus) gradient of U over r, i.e. its derivative over r.

The same results may be also obtained by the proper generalization of the image charge method.

Problem 6.4 (10 points). A uniform electric field $\mathbf{E}_0$ has been created (by external, remote sources) inside a uniform, linear dielectric. Find the changes of field $\mathbf{E}$, created by cutting out a cavity in the shape of a round cylinder of radius R, with the axis perpendicular to the external field (see Fig. on the right).



Solution: Introducing the usual polar coordinates, we can use the general solution (2.112) of the Laplace equation, and our experience with using it for the problem shown in Fig. 2.13 of the lecture notes, as the guidance for looking for the electrostatic potential ϕ in the following form:

$$\phi\Big|_{\rho\le R} = a_1\rho\cos\varphi, \quad \phi\Big|_{\rho\ge R} = (-E_0\rho + \frac{b_1}{\rho})\cos\varphi,$$

where coefficient a_1 has the sense of the (minus) uniform field inside the cavity. Using the boundary conditions of continuity of ϕ and $\varepsilon\partial\phi/\partial n$ (i.e. $\varepsilon_r\partial\phi/\partial\rho$) on the cavity surface ($\rho = R$), we get two equations,

$$a_1 R = -E_0 R + \frac{b_1}{R}, \quad a_1 = \varepsilon_r\left(-E_0 - \frac{b_1}{R^2}\right),$$

for two unknown coefficients, a_1 and b_1. Solving them, we get:

$$a_1 = -\frac{2\varepsilon_r}{\varepsilon_r + 1}E_0, \quad b_1 = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1}E_0 R^2.$$

As a result, the electrostatic potential distribution may be presented as

$$\phi\Big|_{\rho\le R} = -\frac{2\varepsilon_r}{\varepsilon_r+1}E_0\rho\cos\varphi = -\frac{2\varepsilon_r}{\varepsilon_r+1}E_0 x,$$

$$\phi\Big|_{\rho\ge R} = -E_0\left(\rho+\frac{\varepsilon_r-1}{\varepsilon_r+1}\frac{R^2}{\rho}\right)\cos\varphi = -E_0 x\left(1+\frac{\varepsilon_r-1}{\varepsilon_r+1}\frac{R^2}{x^2+y^2}\right),$$

where x is the Cartesian coordinate along the initial field. As a (necessary :-) sanity check, at $\varepsilon_r = 1$ (uniform space with no dielectric), the potential distribution is the same at both $\rho > R$ and $\rho < R$:

$$\phi = \phi_0 = -E_0\rho\cos\theta,$$

and corresponds to the uniform field $E_0\mathbf{n}_x$. In the general case, $\varepsilon_r \neq 1$, the electric field, $\mathbf{E} = -\nabla\phi$, is:

$$\mathbf{E}\Big|_{\rho<R} = \frac{2\varepsilon_r}{\varepsilon_r+1}E_0\mathbf{n}_x, \quad \mathbf{E}\Big|_{\rho>R} = E_0\mathbf{n}_x + \frac{\varepsilon_r-1}{\varepsilon_r+1}E_0\left(\mathbf{n}_x\frac{R^2(y^2-x^2)}{\left(x^2+y^2\right)^2} - \mathbf{n}_y\frac{2R^2xy}{\left(x^2+y^2\right)^2}\right).$$

From here, the electric field change from its original value $E_0\mathbf{n}_x$ is:

$$\Delta\mathbf{E}\Big|_{\rho<R} = \left(\frac{2\varepsilon_r}{\varepsilon_r+1}-1\right)E_0\mathbf{n}_x = \frac{\varepsilon_r-1}{\varepsilon_r+1}E_0\mathbf{n}_x, \quad \Delta\mathbf{E}\Big|_{\rho>R} = \frac{\varepsilon_r-1}{\varepsilon_r+1}E_0\left(\mathbf{n}_x\frac{R^2(y^2-x^2)}{\left(x^2+y^2\right)^2} - \mathbf{n}_y\frac{2R^2xy}{\left(x^2+y^2\right)^2}\right).$$

It is curious that in the limit $\varepsilon_r \to \infty$, the internal electric field *increases* by exactly $E_0\mathbf{n}_x$, i.e. 100% of its initial value, while **D** *drops* dramatically (by factor ε_r). Students are invited to interpret this fact in the light of the thin-gap "experiments" shown in Fig. 3.6 of the lecture notes.