PHY 506 **Classical Electrodynamics II** Spring 2011

**Homework 7 with solutions**

**Problem 7.1** (10 points). Prove that the following quantities: (i) $E^2 - c^2B^2$, and (ii) $\mathbf{E}\cdot\mathbf{B}$ are Lorentz-invariant.

*Solution*: Both facts can be proved by writing these expressions in Cartesian components,

$$E^2 - c^2B^2 = \left(E_x^2 + E_y^2 + E_z^2\right) - c^2\left(B_x^2 + B_y^2 + B_z^2\right), \quad \mathbf{E}\cdot\mathbf{B} = E_xB_x + E_yB_y + E_zB_z,$$

and then using Eqs. (9.134) of the lecture notes for the transformation of each component.

A shorter way to reach the same goal is to notice that according to Eqs. (9.125) and (9.131) of the lecture notes,[1]

$$E^2 - c^2B^2 = -\frac{c^2}{2}F_{\alpha\beta}F^{\alpha\beta}, \quad \mathbf{E}\cdot\mathbf{B} = \frac{c}{4}F_{\alpha\beta}G^{\alpha\beta},$$

so that both combinations are double scalar products, and hence Lorentz-invariant.

**Problem 7.2** (10 points). Find the trajectory of a relativistic particle in a uniform electrostatic field $\mathbf{E}$ for the case of arbitrary initial velocity $\mathbf{u}(0)$.

*Hint*: You are encouraged to explore alternative ways of integration the equation of motion, different from the one used in class for case $\mathbf{u}(0) \perp \mathbf{E}$.

*Solutions*:

(i) An elegant alternative way to solve this problem is to integrate the 4-vector equation (9.145),

$$\frac{dp^\alpha}{d\tau} = qF^{\alpha\beta}u_\beta,$$

directly, considering the proper time $\tau$ of the particle as an argument. For the nonvanishing components of 4-velocity[2] we get equations

$$\frac{d(\gamma c)}{d(\Gamma\tau)} = \gamma u_z, \quad \frac{d(\gamma u_x)}{d(\Gamma\tau)} = 0, \quad \frac{d(\gamma u_z)}{d(\Gamma\tau)} = \gamma c,$$

where $\Gamma \equiv qE/cm$ is a constant parameter with the reciprocal time ($s^{-1}$) dimensionality. The middle equation is elementary, and yields

$$\gamma u_x = \text{const} = \frac{cu_x(0)}{\left[c^2 - u_x^2(0) - u_z^2(0)\right]^{1/2}} \equiv C.$$

The remaining two equations may be combined (by the additional differentiation of any of them over $\tau$ and substitution of the remaining equation) to give similar second-order differential equations

[1] Actually, the first of this expressions has been discussed in class – see Eq. (9.217).

[2] I am using the same coordinate system choice as discussed in Sec. 9.6 of the lecture notes, with axis $z$ along the electric field, and axis $x$ in the plane of motion, so that $u_y = 0$ for any $\tau$.

$$\frac{d^2}{d\tau^2}(\gamma c) = \Gamma^2(\gamma c), \quad \frac{d^2}{d\tau^2}(\gamma u_z) = \Gamma^2(\gamma u_z),$$

with similar solutions

$$\gamma\, c = A\cosh\Gamma(\tau - \tau_0), \quad \gamma\, u_z = A\sinh\Gamma(\tau - \tau_0). \qquad (*)$$

Constants $A$ and $\tau_0$ may be found from the initial conditions:

$$A = c\left[\frac{c^2 - u_z^2(0)}{c^2 - u_x^2(0) - u_z^2(0)}\right]^{1/2} = c\,\frac{\gamma(0)}{\gamma_z(0)}, \qquad \gamma_z(0) \equiv \frac{1}{\left[1 - u_z^2(0)/c^2\right]^{1/2}}, \quad \tau_0 = -\frac{1}{\Gamma}\operatorname{arctanh}\frac{u_z(0)}{c}.$$

Physically, $\tau_0$ is the proper time at which the particle reaches the lowest value of $z$.

Now we can find the tangent to trajectory as

$$\frac{dz}{dx} = \frac{u_z}{u_x} = \frac{A}{C}\sinh\Gamma(\tau - \tau_0),$$

but in order to integrate this equation we still need to express its right-hand part as a function of either $x$ or $z$. For example, we can find $z(\tau)$, integrating the second of Eqs. (*):

$$\gamma u_z \equiv \gamma\frac{dz}{dt} = \frac{dz}{d\tau} = A\sinh\Gamma(\tau - \tau_0),$$

$$z = A\int_0^\tau \sinh\Gamma(\tau - \tau_0)d\tau = \frac{A}{\Gamma}\left[\cosh\Gamma(\tau - \tau_0) - \cosh\Gamma\tau_0\right].$$

From here,

$$\cosh\Gamma(\tau - \tau_0) = \frac{\Gamma z}{A} + \cosh\Gamma\tau_0 = \frac{\Gamma z}{A} + \frac{\gamma(0)c}{A} = \frac{\Gamma z}{A} + \gamma_z(0),$$

$$\sinh\Gamma(\tau - \tau_0) = \left[\left(\frac{\Gamma z}{A} + \gamma_z(0)\right)^2 - 1\right]^{1/2},$$

and now we can finally find the trajectory:

$$\frac{dz}{dx} = \frac{A}{C}\sinh\Gamma(\tau - \tau_0) = \frac{1}{C}\left[\left(\Gamma z + A\gamma_z(0)\right)^2 + A^2\right]^{1/2},$$

$$x = C\int_0^z \frac{dz}{\left[\left(\Gamma z + A\gamma_z(0)\right)^2 - A^2\right]^{1/2}} = \frac{C}{\Gamma}\left[\operatorname{arccosh}\left(\frac{\Gamma z}{A} + \gamma_z(0)\right) - \operatorname{arccosh}\gamma_z(0)\right],$$

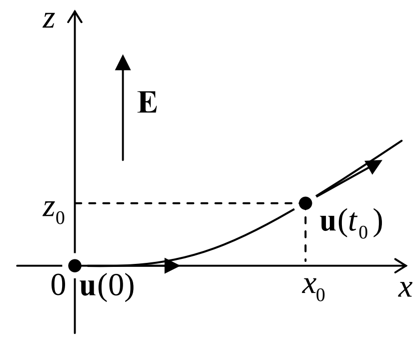
which is a natural generalization of the result derived in Sec. 9.6 for the particular case $u_z(0) = 0$ and hence $\gamma_z(0) = 1$, $A = c\gamma(0)$, $A/\Gamma = \gamma(0)mc^2/qE = \mathcal{E}(0)/qE$, $C/\Gamma = p_0c/qE$, so that

$$x = \frac{p_0 c}{qE}\left[\operatorname{arccosh}\left(\frac{qEz}{\mathcal{E}_0} + 1\right)\right].$$

(ii) Another way to solve this problem is to notice that Eq. (9.165),

$$z = \frac{\mathcal{E}_0}{qE}\left(\cosh\frac{qEx}{cp_0} - 1\right),$$

obtained for the case $\mathbf{u}(0) \perp \mathbf{E}$, may be used in the general case as well, if we shift the origins of $x$, $z$, and $t$ (see Fig. on the right),




$$z = z_0 + \frac{\mathcal{E}(0)}{qE}\left(\cosh\frac{qE(x - x_0)}{cp(0)} - 1\right)$$

and the "only" thing we should express parameters $\mathcal{E}(0)$, $p(0)$, $x_0$, and $z_0$ via components $u_x(0)$, $u_z(0)$of the new initial velocity, which in this notation becomes $\mathbf{u}(t_0)$ However, the recalculation (leading of course to the same result as above) is actually not much easier then the first way.

**Problem 7.3** (15 points). Analyze motion of a nonrelativistic particle in a region where the electric and magnetic fields are both constant and uniform, but not necessarily parallel or perpendicular to each other.

*Solution*: Let us select the direction of vector **B** for axis $z$, and direct axis $y$ so that vector **E** is in the [$y$, $z$] plane see Fig. on the right. Then the nonrelativistic equation of motion (see Eq. (9.144) of the lecture notes with $\gamma_u = 1$, i.e. p = m**u**) has the following Cartesian components:

$$m\ddot{x} = q\dot{y}B, \quad m\ddot{y} = q\left(E_y - \dot{x}B\right), \quad m\ddot{z} = qE_z. \qquad (*)$$

The last equation is independent of its counterparts, and is easily integrated to give the usual "free fall" motion along axis $z$, with constant acceleration $a_z = qE_z/m$. The first two equations may be merged by introduction of the rotational 2D velocity

$$\boldsymbol{u} \equiv \mathbf{n}_x\left(\dot{x} - u_d\right) + \mathbf{n}_y\dot{y}, \quad u_d \equiv \frac{E_y}{B}.$$

Differentiating this vector, and using Eqs. (*),

$$\dot{\boldsymbol{u}} = \mathbf{n}_x\ddot{x} + \mathbf{n}_y\ddot{y} = \mathbf{n}_x\frac{q\dot{y}B}{m} + \mathbf{n}_y\frac{(E_y - \dot{x}B)}{m} = \frac{qB}{m}\left[\mathbf{n}_x\dot{y} - \mathbf{n}_y\left(\dot{x} - u_d\right)\right] = \omega_c\left[\mathbf{n}_x u_y - \mathbf{n}_y u_x\right],$$

we get a differential equation similar to Eq. (9.150) of the lecture notes:

$$\dot{\mathbf{u}}_{\text{rot}} = \omega_c \mathbf{u}_{\text{rot}} \times \mathbf{n}_z,$$

with the cyclotron frequency (9.151): $\omega_c = qB/m$. This means that the particle's projection on plane [$x$, $y$] performs circular cyclotron motion around a point which moves along axis $x$ with a constant drift velocity $u_d = E_y/B$, i.e. essentially the same motion (along a trochoidal trajectory) as for the case of mutually perpendicular fields ($E_y = E$, $E_z = 0$) which was analyzed in the lecture notes – see Fig. 9.11 and its discussion.

Thus, the only essentially new feature resulting from the arbitrary angle between the fields is a accelerated motion of the particle along the direction of vector **B**. This is qualitatively true for a relativistic particle as well, but quantitatively in this case the motion is more complex, because velocity components become coupled via the Lorentz factor $\gamma_u$.

**Problem 7.4** (15 points). Each of two very thin, long, parallel beams of electrons of the same velocity **u** carries electric charge of density $\lambda$ per unit length (as measured in the coordinate frame moving with electrons).

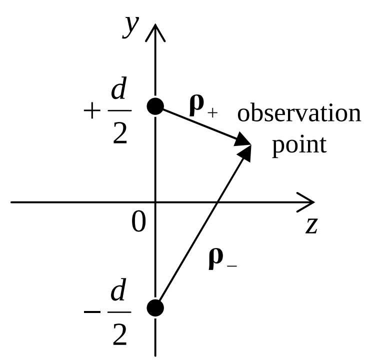
(i) Calculate the distribution of the electric and magnetic fields in the system (outside the beams), as measured in the lab frame.

(ii) Calculate the interaction force between the beams (per particle) and the resulting acceleration, both in the lab frame, and in the system moving with the electrons. Compare the results and give a brief discussion of the comparison.

*Solutions:*

(i) In the reference frame moving with electrons, they are static, so there is no magnetic field: **B**' = 0. The electric field observed in that frame may presented as a sum of two fields (each created by one beam),

$$\mathbf{E}' = \mathbf{E}'_{+} + \mathbf{E}'_{-},$$




and each of these components may be readily found, say, from the Gauss theorem applied to a round cylinder of radius $\rho_\pm$ (see Fig. on the right), with the corresponding beam serving as an axis:

$$\mathbf{E}'_{\pm} = \frac{\lambda \boldsymbol{\rho}_{\pm}}{2\pi\varepsilon_0 \rho_{\pm}^2},$$

or, with the coordinate choice shown in the figure,

$$E'_{\pm x} = 0, \quad E'_{\pm y} = \frac{\lambda (y \mp d/2)}{2\pi\varepsilon_0 \left[ (y \mp d/2)^2 + z^2 \right]}, \quad E'_{\pm z} = \frac{\lambda z}{2\pi\varepsilon_0 \left[ (y \mp d/2)^2 + z^2 \right]}.$$

Now using the Lorentz transform formulas (9.134), in the lab system we also can write

$$\mathbf{E} = \mathbf{E}_{+} + \mathbf{E}_{-},$$

with[3]

$$E_{\pm x} = 0, \quad E_{\pm y} = \gamma_u E'_{\pm y} = \gamma_u \frac{\lambda (y \mp d/2)}{2\pi\varepsilon_0 \left[ (y \mp d/2)^2 + z^2 \right]}, \quad E_{\pm z} = \gamma E'_{\pm z} = \gamma_u \frac{\lambda z}{2\pi\varepsilon_0 \left[ (y \mp d/2)^2 + z^2 \right]},$$

$$B_{\pm x} = 0,$$

$$B_{\pm y} = -\frac{\gamma_u u}{c^2} E'_{\pm z} = -\frac{\gamma_u u}{c^2} \frac{\lambda z}{2\pi\varepsilon_0 \left[ (y \mp d/2)^2 + z^2 \right]}, \quad B_{\pm z} = \frac{\gamma_u u}{c^2} E'_{\pm y} = \frac{\gamma_u u}{c^2} \frac{\lambda (y - d/2)}{2\pi\varepsilon_0 \left[ (y \mp d/2)^2 + z^2 \right]}..$$

3 The Lorentz transform does not change length perpendicular to the relative velocity, so that *y'* = *y*, *z'* = *z*.

(ii) The Lorenz force acting on one beam (say, the one located at $y = +d/2$) comes only from the fields created by the other beam (located at $y' = -d/2$). As a result, in the frame moving with the particles

$$F_y' = qE_{-y}'\Big|_{y=d/2,\, z=0} = \frac{q\lambda}{2\pi\varepsilon_0 d}, \quad F_z' = 0,$$

while in the lab frame the magnetic field contributes to the force as well:

$$\mathbf{F} = q(\mathbf{E}_- + \mathbf{u}\times\mathbf{B}_-)_{y=d/2,\, z=0},$$

and we get

$$F_y = q(E_{-y} - uB_{-z}) = q\gamma_u \frac{\lambda}{2\pi\varepsilon_0 d}\left(1 - \frac{u^2}{c^2}\right) = \frac{1}{\gamma}\frac{q\lambda}{2\pi\varepsilon_0 d}, \quad F_z = q(E_{-z} + uB_{-y}) = 0.$$

The resulting vertical acceleration in the moving frame (where $M' = m$) is

$$a_y' = \frac{F_y'}{m} = \frac{q\lambda}{2\pi\varepsilon_0 dm},$$

while in the lab frame, where $M = \gamma_u m$, it is

$$a_y = \frac{F_y}{M} = \frac{1}{\gamma_u^2}\frac{q\lambda}{2\pi\varepsilon_0 dm} = \frac{a_y'}{\gamma_u^2}.$$

These results for $F_y$ and $a_y$ are dramatically different, but they are actually consistent, because time runs differently in the two frames. For example, if the acceleration produces a small beam shift $\Delta y \ll d$, with the vertical velocity still much below $c$, we can write

$$\Delta y' = \frac{a_y'}{2}(\Delta t')^2 = \frac{a_y'}{2}(\Delta\tau)^2, \quad \Delta y = \frac{a_y}{2}(\Delta t)^2 = \frac{a_y'}{2\gamma^2}(\Delta t)^2.$$

Since according to Eq. (9.21) the "proper" time interval $\Delta t' = \Delta\tau$ and the lab frame interval $\Delta t$ are related as $\Delta t = \gamma_u \Delta\tau$ ("time dilation"), we get that $\Delta y' = \Delta y$, as it should be.