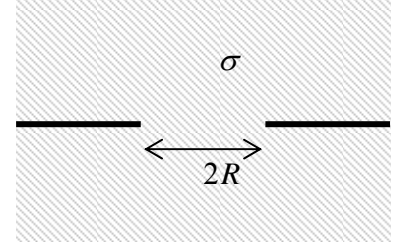


Problem 7.1 (to be graded of 15 points). Calculate resistance between two large conductors separated with a very thin, plane, insulating partition, with a circular hole of radius R in it – see Fig. on the right.



Hint: You may like to use the degenerate ellipsoidal coordinates which have been used in class to find the self-capacitance of a round disk in vacuum – see Sec. 2.3 of the lecture notes

Solution: Selecting parameter R of the degenerate ellipsoidal coordinates given by Eq. (2.59) of the lecture notes, to be equal to the hole radius, and placing the origin into its center, we see that the intact part of the partition corresponds to $\beta = \pi/2 = \text{const}$, and at the conductor points slightly above the partition, $z \approx R \sinh \alpha (\pi/2 - \beta)$, so that

$$\left. \frac{\partial \phi}{\partial n} \right|_{z=0, r>R} = \left. \frac{\partial \phi}{\partial z} \right|_{z=0, r>R} = -\frac{1}{R \sinh \alpha} \frac{\partial \phi}{\partial \beta}.$$

This means that if the potential is a function of α alone, the boundary condition of having no current flow through the partition is automatically satisfied. (Due to the problem symmetry, the same conclusion is valid for the lower semi-space as well.) From the disk capacitance problem (Sec. 2.3 of the lecture notes), we already know that with this assumption, the Laplace equation may be also satisfied if

$$\phi = c_1 \arctan(\sinh \alpha) + c_2.$$

For the sake of notation simplicity, let us accept (anti)symmetric boundary conditions at infinity:

$$\phi \rightarrow \mp \frac{V}{2} \text{sgn}(z) = \mp \frac{V}{2} \text{sgn}(\alpha), \quad \text{when } |z| \rightarrow \infty, \text{ i.e. when } |\alpha| \rightarrow \infty$$

(with applied voltage driving current up along axis z); in this case the solution takes the form

$$\phi = -\frac{V}{\pi} \arctan(\sinh \alpha).$$

Note that according to this formula, the contribution of distant parts of the conductor (with $r \gg R$) into the total voltage drop is negligible; this is why the shape of external electrodes is not important, provided that they are not too small or too close to the hole.

What remains now is to find total current

$$I = \int_A j_n d^2 r = \sigma \int_A \frac{\partial \phi}{\partial n} d^2 r,$$

where A is an arbitrary surface crossed by the whole current flow. (Due to the current continuity, the result does not depend on the choice of the surface.) The easiest choice of such surface is evidently the plane $z = 0$ (more exactly, its fraction inside the hole, $r < R$). Near this plane, $z \approx R \alpha \cos \beta$, while our solution is just $\phi \approx -(V/\pi) \alpha$, so that $-\partial \phi / \partial z = V/R \pi \cos \beta$. Since at this plane, the distance of a point from the center is $\rho \equiv (x^2 + y^2)^{1/2} = R \sin \beta$, we get

$$I = -\sigma \int_{\substack{z=0 \\ \rho < R}} \frac{\partial \phi}{\partial z} d^2 \rho = -\sigma \int_{z=0} \frac{\partial \phi}{\partial z} d^2 \rho = -2\pi\sigma \int_0^R \frac{\partial \phi}{\partial z} \rho d\rho = 2\pi\sigma \frac{V}{\pi} \int_0^R \frac{\rho d\rho}{R \cos \beta} = \sigma V \int_{\rho=0}^{\rho=R} \frac{d(\rho^2)}{\sqrt{R^2 - \rho^2}} = 2\sigma VR,$$

so that the hole resistance V/I is just $1/2\sigma R$.

This important (so-called “Maxwell” or “Maxwell-Rayleigh”) formula is frequently remembered as the ratio of the material resistivity $1/\sigma$ to the hole diameter $2R$, without any numerical coefficients. It may be conveniently interpreted as a resistance of a round cylinder with the cross-section $A = \pi R^2$, and the effective length $L = 2\pi R$, defined by the current spreading effect.

Problem 7.2 (15 points). Calculate the effective (average) conductivity σ_{ef} of a medium with many empty spherical cavities of radius R , carved at random points in a uniform Ohmic conductor (see Fig. on the right), in the limit of small sphere concentration, $n \ll R^{-3}$.

Hint: Try to use the analogy with a dipole media (Sec. 3.2 of the lecture notes)..

Solution: In class, we have found that a single spherical cavity, cut in a conductor with a uniform current flow of density \mathbf{j}_0 , creates outside it an additional dipole field described by the second term in Eq. (4.26):

$$\phi_{\text{add}} = -\frac{j_0}{\sigma} \frac{R^3}{2r^2} \cos \theta.$$

According to Eq. (3.7), this field corresponds to the following dipole moment:

$$\mathbf{p} = -4\pi\epsilon_0 \frac{R^3}{2\sigma} \mathbf{j}_0.$$

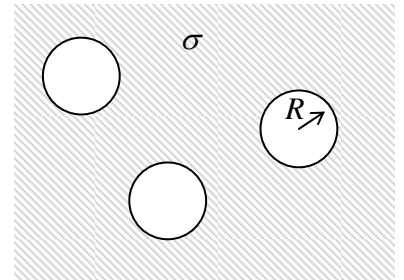
If the concentration of spheres is low ($nR^3 \ll 1$), these dipole fields add up independently, creating the average electric polarization

$$\mathbf{P} = n\mathbf{p} = -4\pi\epsilon_0 \frac{nR^3}{2\sigma} \mathbf{j}_0.$$

Our analysis of electrically polarized media in Sec. 3.2 has shown that the polarization (divided by ϵ_0) is effectively subtracted from the field created by external charges. In particular, Eq. (3.28) may be rewritten as

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P}}{\epsilon_0}, \quad \mathbf{E}_0 \equiv \frac{\mathbf{D}}{\epsilon_0},$$

where \mathbf{E} is the average electric field, and vector \mathbf{D} is defined by external charge density ρ via Eq. (3.27). In the case of a conductor, the initial field \mathbf{E}_0 is created by an external e.m.f., and equals to \mathbf{j}_0/σ . However, all the arguments concerning the effect of polarization did not use any assumptions about the media, and may be used for conductors as well. This means that in our case the average field is



$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P}}{\epsilon_0} = \frac{\mathbf{j}_0}{\sigma} + 4\pi \frac{nR^3}{2\sigma} \mathbf{j}_0 \equiv \frac{\mathbf{j}_0}{\sigma_{\text{ef}}},$$

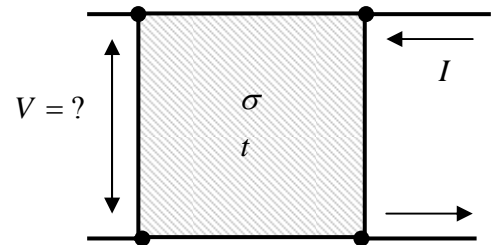
where σ_{ef} is the effective conductance we were looking for:

$$\sigma_{\text{ef}} = \frac{\sigma}{1 + 2\pi nR^3} \approx \sigma(1 - 2\pi nR^3) \leq \sigma.$$

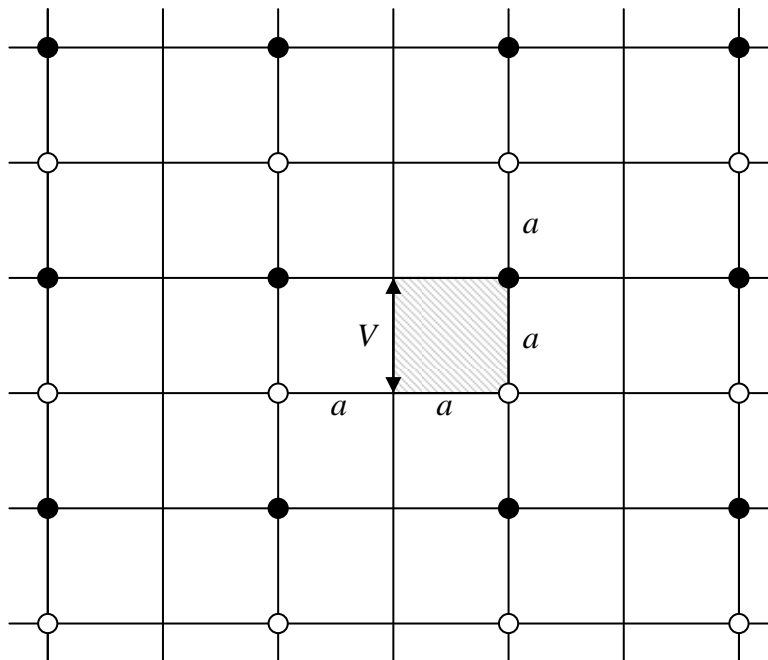
The last (approximate) equality is valid because our analysis is limited to the small cavity concentrations, $nR^3 \ll 1$, when the correction to σ is small. Of course, the reduction of conductivity, i.e. the sign of the difference $\Delta\sigma < 0$ due to carving out the cavities could be predicted from handwaving arguments, and so could be the proportionality of the ratio $\Delta\sigma/\sigma \ll 1$ to nR^3 , so that the main result of our calculation is the particular (and quite large) numerical coefficient 2π .

Problem 7.3 (15 points). Find voltage drop V between two corners of a square cut from a uniform sheet with Ohmic conductivity, induced by dc current I which is passed between its two other corners (see Fig. on the right).

Hint: Try to use the dc current analog of the charge image method.



Solution: As has been discussed in Sec. 4.3 of the lecture notes, we can replace a straight conductor edge with an additional current injection point, at the mirror reflection point. Following this approach, and taking into account all 4 sides of the square, we see that the original problem may be replaced with an infinite conducting sheet with a periodic mesh of injection/ejection points shown in Fig. below. (Here a solid dot means the injection of current $4I$, while on open point, the ejection of a similar current.)



One can readily check that the net current, which is a vector sum of currents radially spreading from/into each injection/ejection point, does not cross any border of the real sheet square (shown by hatching). In our 2D case, each of these currents creates current density

$$\mathbf{j}_k(\boldsymbol{\rho}) = \frac{I_k}{2\pi t} \frac{(\boldsymbol{\rho} - \boldsymbol{\rho}_k)}{|\boldsymbol{\rho} - \boldsymbol{\rho}_k|^2} = \pm \frac{2I}{\pi t} \frac{(\boldsymbol{\rho} - \boldsymbol{\rho}_k)}{|\boldsymbol{\rho} - \boldsymbol{\rho}_k|^2},$$

and hence the following electric field,

$$\mathbf{E}_k(\boldsymbol{\rho}) = \frac{\mathbf{j}_k(\boldsymbol{\rho})}{\sigma} = \pm \frac{2I}{\pi t \sigma} \frac{(\boldsymbol{\rho} - \boldsymbol{\rho}_k)}{|\boldsymbol{\rho} - \boldsymbol{\rho}_k|^2},$$

which may be presented as the (minus) gradient of the following electrostatic potential:

$$\phi_k(\boldsymbol{\rho}) = \mp \frac{2I}{\pi t \sigma} \ln|\boldsymbol{\rho} - \boldsymbol{\rho}_k| + \text{const} = \mp \frac{I}{\pi t \sigma} \ln\left[|\boldsymbol{\rho} - \boldsymbol{\rho}_k|^2\right] + \text{const}.$$

The integration constant in the last expression is not important if we are interested only in voltage V (i.e. potential difference) between the points, and we can take it for zero.

For our points of interest (square corners),

$$|\boldsymbol{\rho} - \boldsymbol{\rho}_k|^2 = a^2[(2k+1)^2 + k'^2],$$

where k and k' are the integers listing, respectively, the horizontal and vertical distances (in units of a) between the corner and the current injection/ejection points. Now using the linear superposition principle to sum up the contributions of all the points (see Fig. above), we get

$$V = \frac{2I}{\pi t \sigma} \sum_{k, k'=-\infty}^{+\infty} \ln \frac{(2k+1)^2 + (2k'+1)^2}{(2k+1)^2 + (2k')^2} \approx 0.72 \frac{I}{t \sigma}.$$

This result shows that the voltage is somewhat lower than the value $I/t\sigma$ that we would get from the uniformly-distributed injection of the current into opposite (in our Fig., horizontal) sides of the square sheet. This is natural, because at the corner injection used in our problem, most current passes close to the side connecting the electrodes, thus reducing the voltage drop across the opposite side.