

Problem 8.1 (10 points). A charged, relativistic particle has been injected into a uniform electric field which oscillates in time with frequency ω . Calculate the time dependence of the particle's velocity (as observed from the lab frame).

Solution: Directing axis x along the electric field, and axis y perpendicular to it, but within the plane of the initial velocity of the particle (so that $\beta_z \equiv 0$), we can write the relativistic equation (9.144) of its motion, in the lab frame, as

$$\frac{d(\gamma\beta_x)}{d(\omega t)} = \kappa \cos \omega t, \quad \frac{d(\gamma\beta_y)}{d(\omega t)} = 0, \quad \text{where } \kappa \equiv \frac{qE_\omega}{mc\omega},$$

E_ω being the electric field amplitude. These equations may be readily integrated:

$$\gamma\beta_x = \gamma_0\beta_{x0} + \kappa \sin \omega t, \quad \gamma\beta_y = \gamma_0\beta_{y0}. \quad (*)$$

Since

$$\gamma^2 \equiv \frac{1}{1-\beta^2} = \frac{1}{1-\beta_x^2 - \beta_y^2}, \quad \text{i.e. } \gamma^2(1-\beta_x^2 - \beta_y^2) = 1, \quad \text{i.e. } \gamma^2 = 1 + (\gamma\beta_x)^2 + (\gamma\beta_y)^2,$$

we can now calculate the Lorentz parameter as a function of time:

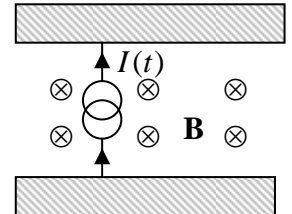
$$\gamma = \left[1 + (\gamma\beta_x)^2 + (\gamma\beta_y)^2 \right]^{1/2} = \left[1 + (\gamma_0\beta_{x0} + \kappa \sin \omega t)^2 + (\gamma_0\beta_{y0})^2 \right]^{1/2}. \quad (**)$$

Now we can combine Eqs. (*) and (**) to find both reduced velocities as explicit functions of time:

$$\beta_x = \frac{\gamma_0\beta_{x0} + \kappa \sin \omega t}{\left[1 + (\gamma_0\beta_{x0} + \kappa \sin \omega t)^2 + (\gamma_0\beta_{y0})^2 \right]^{1/2}}, \quad \beta_y = \frac{\gamma_0\beta_{y0}}{\left[1 + (\gamma_0\beta_{x0} + \kappa \sin \omega t)^2 + (\gamma_0\beta_{y0})^2 \right]^{1/2}},$$

$$\beta \equiv \left[\beta_x^2 + \beta_y^2 \right]^{1/2} = \left[\frac{(\gamma_0\beta_{x0} + \kappa \sin \omega t)^2 + (\gamma_0\beta_{y0})^2}{1 + (\gamma_0\beta_{x0} + \kappa \sin \omega t)^2 + (\gamma_0\beta_{y0})^2} \right]^{1/2}.$$

Problem 8.2 (10 points). Consider the simple model of capacitor charging shown in Fig. on the right, and prove that the momentum given by a uniform, stationary magnetic field \mathbf{B} to the current-carrying conductor is equal and opposite to the momentum of the EM field which the current $I(t)$ builds up in the capacitor. (You may let the capacitor be planar and very broad, and neglect the fringe field effects.)



Solution: In the geometry shown in the figure, with positive $I(t)$, the total magnetic force (5.8) exerted on the wire is directed to the left and has magnitude $F(t) = I(t)Bd$, where d is the wire length, i.e. the distance between capacitor's plates. Integrating this expression over time, for momentum transferred from the field to the wire we get

$$p_{\text{wire}} = \int F(t)dt = BdQ, \quad (*)$$

where $Q \equiv \int I(t)dt$ is the capacitor's charge built up by the current.

If $Q > 0$, is positive, the electric field it creates is directed, in our figure, from the top down, so that the EM field's momentum density vector (9.237) is directed to the right, i.e. in the opposite direction. The total field momentum has magnitude

$$p_{\text{field}} = \int g d^3 r = Adg = Ad \frac{S}{c^2} = Ad \frac{EH}{c^2} = Ad \frac{(V/d)(B/\mu_0)}{c^2} = \frac{\epsilon_0 A}{d} VBd.$$

Here A is the capacitor's area, and $V = Ed$ is the accumulated voltage between its plates. According to Eq. (2.28), the fraction in the last expression is just the mutual capacitance C , so that its product by V equals the same Q which participates in Eq. (*). Thus two momenta, p_{wire} and p_{field} are indeed equal and opposite.

Problem 8.3 (10 points). Calculate the pressure imposed on well-conducting walls of a waveguide with rectangular ($a \times b$) cross-section by a wave propagating along it in the fundamental (H_{10}) mode. Give an interpretation of the result.

Solution: According to Eqs. (7.127), (7.133), (7.134) of the lecture notes, with $n = 1$ and $m = 0$, the broader walls of the resonator (in Fig. 7.21, with $y = 0, b$) are exposed to both the electric and magnetic fields of the wave, with complex amplitudes

$$E_x = 0, \quad E_y = i \frac{ka}{\pi} ZH_l \sin \frac{\pi x}{a}, \quad E_z = 0, \quad H_x = -i \frac{k_z a}{\pi} H_l \sin \frac{\pi x}{a}, \quad H_y = 0, \quad H_z = H_l \cos \frac{\pi x}{a},$$

while at the narrower walls ($x = 0, b$), only the longitudinal magnetic field is nonvanishing, with complex amplitude equal to $\pm H_l$. Wave vector components in these formulas are related by Eqs. (7.118), (7.128), which in our case ($k_x = \pi/a, k_y = 0$) may be rewritten as

$$\left(\frac{ka}{\pi} \right)^2 - \left(\frac{k_z a}{\pi} \right)^2 = 1.$$

Plugging these relations into Eqs. (9.240) and (9.242), for the average pressure on the walls in terms of the longitudinal field amplitude H_l :

$$\overline{\frac{dF}{dA}} \Big|_{y=0,b} = \frac{\mu_0}{4} |H_l|^2 \left(\cos^2 \frac{\pi x}{a} - \sin^2 \frac{\pi x}{a} \right), \quad \overline{\frac{dF}{dA}} \Big|_{x=0,a} = \frac{\mu_0}{4} |H_l|^2,$$

where time averaging gave an extra factor $1/2$. Pressure on the narrower walls is positive, due to magnetic field effect, while that on the broader walls is negative in the middle, due to the electric field effect. On the average, these two pressures compensate each other, so that the net forces exerted on the broader walls are equal zero.

These results may be readily explained in terms of the fundamental relation between the flux $c\mathbf{g} = \mathbf{S}/c$ of the wave's momentum and its pressure on the mirrors which reflects them. Indeed, the H_{10} wave may be presented as a sum of two plane waves, propagating with wave vectors $\mathbf{k}_{\pm} = \{\pm\pi/a, 0, k_z\}$ which are reflected from the narrower walls of the waveguide – see Homework Problem 2.2. These reflections create an outward pressure on the walls, similar to that at the normal incidence – see Eq. (9.247) of the lecture notes and its discussion. On the other hand, since for H_{10} waves $k_y = 0$, they are not reflected from broader walls of the waveguide, and as a result do not exert a net force on such a wall.