PHY 505 **Classical Electrodynamics I** Fall 2010

Homework 8 with solutions

Problem 8.1 (to be graded of 10 points). Calculate the magnetic field distribution along the axis of a straight solenoid (Fig. 5.6a of the lecture notes) with a finite length *l*, and a round cross-section of radius *R*. Assume that the solenoid has many wire turns ($N >> 1$) which are uniformly distributed along its length.

Solution: Because of the large number of wire turns, and hence the almost circular shape of each turn, the solenoid field may be presented as a sum of field of *N* circular current loops. The field induced by a single loop has been calculated in class - see Eq. (5.23) of the lecture notes. Generalizing it to the case of arbitrary position (*z'*) of the loop at the *z*-axis, we get

$$B_1 = \frac{\mu_0 I}{2} \frac{R^2}{\left[R^2 + (z-z')^2\right]^{3/2}} .$$

Now we can find induced by the solenoid as a whole as

$$B_N = \frac{\mu_0 I}{2} \sum_{j=-N/2}^{N/2} \frac{R^2}{\left[R^2 + (z-z_j)^2\right]^{3/2}} = \frac{\mu_0 I}{2} \sum_{j=-N/2}^{+N/2} \frac{R^2}{\left[R^2 + (z-jl/N)^2\right]^{3/2}} ,$$

where index *j* numbers wire turns (starting from the middle of the solenoid). For $N >> 1$ this sum is equivalent to integral[1]

$$B_N = \frac{\mu_0 I}{2} \int_{-N/2}^{+N/2} \frac{R^2}{\left[R^2 + (z-jl/N)^2\right]^{3/2}} dj = \frac{\mu_0 I}{2} \frac{N}{l} \int_{-l/2}^{+l/2} \frac{R^2}{\left[R^2 + (z-z')^2\right]^{3/2}} dz'$$

$$= \frac{\mu_0 I}{2} \frac{N}{l} \left[\frac{z-z'}{\sqrt{R^2 + (z-z')^2}} \right]_{z'=-l/2}^{z'=+l/2} = \frac{\mu_0 I}{2} \frac{N}{l} \left[\frac{z-l/2}{\sqrt{R^2 + (z-l/2)^2}} - \frac{z+l/2}{\sqrt{R^2 + (z+l/2)^2}} \right].$$

Well inside a long solenoid ($l >> |z|, R$), this expression reduces to Eq. (5.40). On the other hand, at $|z| >> R, l$, this result gives

$$B_N \to \frac{\mu_0 INR^2}{2|z|^3} = \frac{\mu_0}{4\pi} \frac{2m_N}{|z|^3}, \quad m_N = \pi INR^2 = Nm_1 ,$$

i.e. the dipole field, with the dipole moment *N* times larger than that of one current loop.

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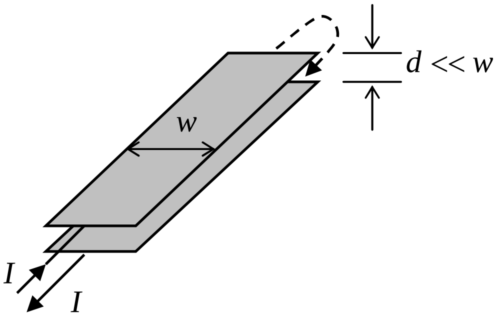
[1] Please notice this most prudent way for the transfer to the continuous limit. Always use it when you are in doubt.

Problem 8.2 (10 points). Two parallel, uniform, long strips of thin foil, separated by distance $d \ll w$, are used as a current loop (see Fig. on the right) . Calculate:

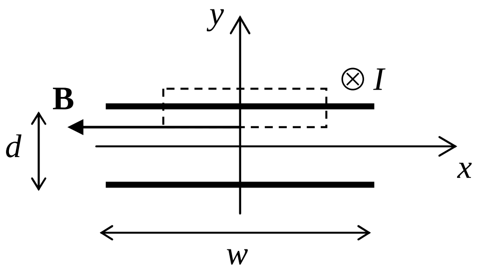
(i) the distribution of the magnetic field and vector-potential,

(ii) the magnetic force (per unit length) acting on each strip, and

(ii) the magnetic energy and self-inductance of the system (per unit length).



Solutions:



(i) Due to the condition $d \ll w$, the magnetic field is localized, and is uniform, in the gap between the strips. Applying the Ampère law to the contour shown with the dashed line in Fig. on the right (which presents the cross-section of the system), we get

$$B = -B_x = \mu_0 \frac{I}{w}. \qquad (*)$$

According to Eq. (5.28) of the lecture notes, the vector-potential has to be directed along axis z, i.e. along the current, and be independent of z. From the structure of that equation, it is also clear that at $d \ll w$, **A** at points well inside the gap should not depend on x either (because the strip edges are not "visible" from those points). Hence we may look for the vector-potential in the form

$$\mathbf{A} = A(y)\mathbf{n}_z .$$

Calculating the curl of such a vector,

$$\nabla \times \mathbf{A} = \mathbf{n}_x \frac{\partial A}{\partial y},$$

and requiring that it is equal to the vector **B** described by Eq. (*), we get

$$\mathbf{A} = -\mu_0 \frac{I}{w} y \mathbf{n}_z + \text{const.}$$

This result, of course, could be also obtained by the direct integration of Eq. (5.28) along x' and z'. (Alternatively, we can integrate Eq. (5.51), written for each current component $dI = Idx'/w$, along axis x'). Notice, however, that a uniform field, like our $\mathbf{B} = -B\mathbf{n}_x$, cannot "tell" one transversal coordinate (say, y) from another one (z), and hence may be also "generated" by different vector-potential distributions, for example[2]

$$\mathbf{A} = \mu_0 \frac{I}{w} z \mathbf{n}_y + \text{const},$$

or a linear combination of these two functions. This is one more manifestation of the gauge invariance of the magnetic field with respect to any transformation described by Eq. (5.45).

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[2] Notice that this distribution, of course, does *not* satisfy Eq. (5.28).

(ii) The total magnetic force acting on each strip is of course directed along axis y, and corresponds to strip repulsion. In order to calculate its magnitude, it would be wrong to plug Eq. (*) into Eq. (5.15), because a current (like a charge) does exert force on itself.[3] As we can readily check from the Ampère law, the field created by a single strip is twice lower:

$$B_1 = \mu_0 \frac{I}{2w} ,$$

so that, according to Eq. (5.15), the force magnitude (per unit length)

$$\frac{F}{l} = \frac{1}{l} \int_{\text{strip}} jB_1 d^3r = IB_1 = \mu_0 \frac{I^2}{2w}.$$

This expression may be also presented as the integral, over the strip area, of the positive pressure of the magnetic field:

$$P \equiv \frac{F}{lw} = \frac{\mu_0 I^2}{2w^2} = \frac{B^2}{2\mu_0} = u ,$$

where $u = U/lwd = \mathrm{U}/V_{\text{gap}}$ is the magnetic energy density – see Eq. (5.78).

(iii) Since the full magnetic field is uniform inside the gap (with the cross-section area dw) and vanishes outside of it, the magnetic energy per unit length is just[4]

$$\frac{U}{l} = \frac{1}{l} \int_{gap} \frac{B^2}{2\mu_0} d^3r = \frac{B^2}{2\mu_0} dw = \mu_0 \frac{d}{w} \frac{I^2}{2} = u \frac{V_{\text{gap}}}{l} .$$

From this result and Eq. (5.79) we immediately get the following expression for the self-inductance (also per unit length):[5]

$$\frac{L}{l} = \mu_0 \frac{d}{w} .$$

This simple formula, which shows a clear way for the reduction of current loop inductances (bring the counterpart conductors as close to each other as possible!), is very important for applications, because such inductances frequently play a negative role in high-speed integrated circuits, reducing interconnect bandwidth.

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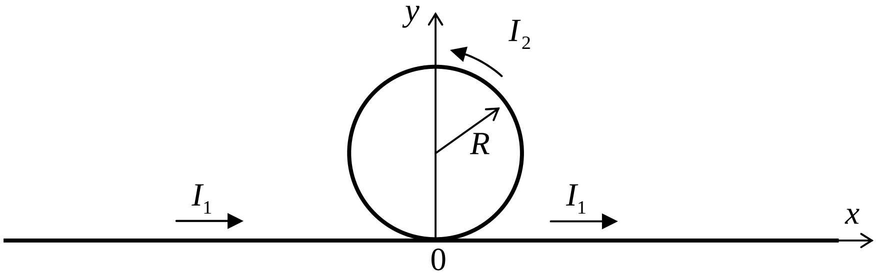
[3] Of course, each component jdx of the current distributed in a strip does exerts some force on each other component jdx' of current in the same strip, but this force is directed along axis x, rather than y.

[4] This result poses an additional question for the reader: how can the strip *repulsion* (i.e. their "desire" to *increase* d, and hence volume V_{gap}, and hence product $U = uV_{\text{gap}}$) be compatible with the apparent general trend for any system to *decrease* its potential energy. As a reminder, for the *electrostatic* force exerted on conductor this paradox does not exits, because it corresponds to *negative* pressure – see, e.g., the model solution of optional Problem O.3.

[5] The same result may be calculated as the ratio $(\Phi/l)/I$, where Φ is the magnetic flux in the gap between the strips:

$$\frac{\Phi}{l} = \frac{1}{l} \int_{gap} B_n d^2r = Bt = \mu_0 \frac{I}{w} t .$$

Problem 8.3 (10 points). Find the mutual inductance of a long straight wire and a round wire loop adjacent to it (see Fig. below). Neglect the thickness of both wires.



Solution: In the Cartesian coordinates shown in Fig. above, Eq. (5.20) of the lecture notes yields

$$B_1 = \frac{\mu_0 I_1}{2\pi\, y},$$

giving the following magnetic flux through the round wire loop:

$$\Phi_{21} = \frac{\mu_0 I_1}{2\pi}\int_{-R}^{+R} dx \int_{R-\sqrt{R^2-x^2}}^{R+\sqrt{R^2-x^2}} dy \frac{1}{y} = \frac{\mu_0 I_1}{\pi}\int_0^R \ln\frac{R+\sqrt{R^2-x^2}}{R-\sqrt{R^2-x^2}}\,dx = \frac{\mu_0 I_1 R}{\pi}\int_0^1 \ln\frac{1+\sqrt{1-\xi^2}}{1-\sqrt{1-\xi^2}}\,d\xi.$$

This is a table integral equal to π,[6] so that

$$\Phi_{21} = \mu_0 I_1 R,$$

and the final answer for the mutual inductance $M = L_{12} = L_{21} = \Phi_{21}/I_1$ is extremely simple:

$$M = \mu_0 R\,.$$

Note that in contrast with the finite mutual inductance, *self*-inductances of both wires are formally infinite in the thin-wire approximation, though this divergence is very weak:

$$L \sim \mu_0 \times (\text{wire length}) \times \ln\frac{(\text{wire length})}{(\text{wire cross-section's radius})}\ .$$

Calculation of L for a round loop made of a wire with small but still nonvanishing cross-section radius $r << R$ is a very useful exercise which is highly recommended to the reader as an additional problem.

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[6] See, e.g., MA Eq. (6.11).