PHY 506

Classical Electrodynamics II

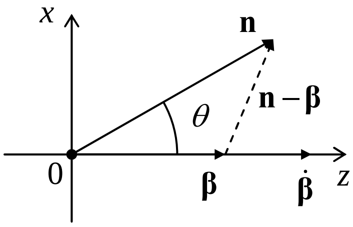
Spring 2011

Homework 9 with solutions

Problem 9.1 (10 points). Analyze the polarization of the EM radiation of a linearly accelerated relativistic particle.

Solution: Let us consider the expression for the electric field of the radiation, given by the second term of Eq. (20a) of the lecture notes:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 cR}, \qquad (*)$$



where all variables have to be taken in the retarded point. Using the coordinate choice shown in Fig. 10.4a (reproduced on the right), in which the observation point **r** = R**n** is within plane [x, z], we see that vectors (**n** - **β**) and $\dot{\boldsymbol{\beta}}$ are also within this plane. Hence the internal vector product in Eq. (*) is perpendicular to that plane (antiparallel to axis y). As a result, vector **E** is in plane [x, z] of the drawing, and perpendicular to the direction **n** toward the radiation observer. Thus we conclude that the radiation is linearly polarized, with vector **E** in the common plane of the observation point and particle's trajectory.

Problem 9.2 (10 points). Find the time dependence of the kinetic energy $\mathcal{E}$ of a charged relativistic particle performing synchrotron motion in a constant and uniform magnetic field **B**, and hence emitting synchrotron radiation. Sketch particle's trajectory.

Hint: You may assume that the energy loss is relatively slow ($-d\mathcal{E}/dt \ll \omega_c\mathcal{E}$), but should spell out the condition of validity of this assumption.

Solution: Using Eq. (9.153) of the lecture notes for the cyclotron orbit's radius R,

$$R = \frac{m\gamma u}{qB} \equiv \frac{mc}{qB}\beta\gamma, \qquad (**)$$

we can rewrite Eq. (10.43) for radiative energy loss as

$$-\frac{d\mathcal{E}}{dt} = \frac{Z_0 q^4 B^2}{6\pi m^2}\beta^2\gamma^2 = \frac{Z_0 q^4 B^2}{6\pi m^2 c^4}\left(\gamma^2 - 1\right) = \frac{1}{2mc^2\tau}\left[\mathcal{E}^2 - \left(mc^2\right)^2\right], \quad \text{where } \tau \equiv \frac{3\pi m^3 c^2}{Z_0 q^4 B^2}.$$

Separating variables of this differential equation,

$$\frac{d\varepsilon}{\varepsilon^2 - 1} = -\frac{dt}{2\tau}, \quad \text{where } \varepsilon \equiv \frac{\mathcal{E}}{mc^2} \geq 1,$$

we can readily integrate it:[1]

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[1] As a math reminder, the integral in the left-hand part may be reduced to two integrals of the type $\int d\xi/\xi = \ln\xi$ via presenting the involved fraction as a sum of two simple poles:

$$\frac{1}{\varepsilon^2 - 1} = \frac{1}{2(\varepsilon - 1)} - \frac{1}{2(\varepsilon + 1)}.$$

$$\frac{1}{2}\ln\frac{\varepsilon-1}{\varepsilon+1} = -\frac{t}{2\tau} + \text{const}\,.$$

Solving for ε, we finally get

$$\mathcal{E} = mc^2 \coth\left(\frac{t}{\tau} + \text{const}\right).$$

This expression shows that the kinetic energy $\mathcal{E} - mc^2$ of the particle is gradually lost on the time scale given by constant τ. (For an electron in a typical magnetic field of 1 T, this constant is close to 2.5 seconds.) The above theory, which assumes that each orbit is circular, is only valid if $\omega_c\tau >> 1$. Rewriting Eq. (9.151) as

$$\omega_c = \frac{qB}{\gamma m},$$

we can present the validity condition as

$$\gamma << \frac{m^2c^2}{Z_0 q^3 B};$$

for the above example, the right-hand part of this relation is of the order of 10^{12}, so that is all real systems it is well satisfied.

As the kinetic energy goes down due to radiative looses, so does parameter $\beta\gamma$, so that, as Eq. (*) shows, the cyclotron radius decreases as well. Hence particle's trajectory is a converging spiral.

Problem 9.3 (20 points). Calculate the power spectrum of the intensity of radiation emitted by a relativistic particle, performing harmonic 1D oscillations, in a certain direction.

Solution: The radiation induced by this periodic motion is also evidently periodic, so that its spectrum may only consist of harmonics of the motion frequency ω_0. Their amplitudes may be recovered from Eq. (10.61) of the lecture notes (which has been derived with the continuous spectrum in mind) by careful manipulation with delta-functions, but it is simpler to recast all basic equations of the spectral theory, developed in Sec. 10.3, for the periodic time dependence of the radiated field. In particular, Eq. (10.51) needs to be changed for a sum over integer harmonic numbers (say, m):

$$\mathbf{E}(t) = \sum_{m=-\infty}^{+\infty} \mathbf{E}_m e^{-im\omega_0 t}\,.$$

The magnetic field may be expanded into a similar series, with complex amplitudes $\mathbf{H}_n = \mathbf{n}\times\mathbf{E}_n/Z_0$, so at time averaging (over the period $\Delta t = 2\pi/\omega_0$ of the motion), the Poynting vector falls into the sum of independent harmonic components:

$$\overline{S_n} = \overline{E(t)H(t)} = \frac{1}{Z_0}\sum_{m,m'=-\infty}^{+\infty} E_m E_{m'}\overline{e^{-i(m+m')\omega_0 t}} = \frac{1}{Z_0}\sum_{m,m'=-\infty}^{+\infty} E_m E_{m'}\delta_{m+m',0} = \frac{1}{Z_0}\sum_{m=-\infty}^{+\infty} E_m E_{-m} = \frac{2}{Z_0}\sum_{m=1}^{+\infty}\left|\mathbf{E}_m\right|^2\,.$$

Thus instead of Eq. (10.54), we can immediately write the following expression for the average radiation power per unit solid angle:

$$\frac{d\overline{\mathscr{P}}}{d\Omega}=\sum_{m=1}^{\infty}\frac{d\overline{\mathscr{P}_m}}{d\Omega},\quad \frac{d\overline{\mathscr{P}_m}}{d\Omega}=\frac{8\pi R^2}{Z_0}\left|\mathbf{E}_m\right|^2 \quad .$$

Into the last expression we have to plug in the Fourier amplitude $\mathbf{E}_m$ from the reciprocal Fourier transform,

$$\mathbf{E}_m=\frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2}\mathbf{E}(t)e^{im\omega_0 t}dt=\frac{\omega_0}{2\pi}\int_{\pi/\omega_0}^{+\pi/\omega_0}\mathbf{E}(t)e^{im\omega_0 t}dt,\quad \text{so that}\quad \frac{d\overline{\mathscr{P}_m}}{d\Omega}=\frac{2R^2\omega_0^2}{\pi Z_0}\left|\int_{\pi/\omega_0}^{+\pi/\omega_0}\mathbf{E}(t)e^{im\omega_0 t}dt\right|^2 .$$

Now plugging $\mathbf{E}(t)$ from Eq. (10.20a) and repeating the transformation from integration over t to that over t', instead of Eq. (10.61) we get

$$\frac{d\overline{\mathscr{P}_m}}{d\Omega}=\frac{Z_0q^2m^2\omega_0^4}{8\pi^3}\left|\int_{-\pi/\omega_0}^{+\pi/\omega_0}\mathbf{n}\times\left[\mathbf{n}\times\boldsymbol{\beta}(t')\right]\exp\left\{im\omega_0\left(t'-\frac{\mathbf{n}\cdot\mathbf{r}'}{c}\right)\right\}dt'\right|^2 .$$

Now returning to our particular problem, and selecting the coordinates shown in Fig. 10.4a of the lecture notes, we can write

$$\mathbf{n}=\{\sin\theta,\,0,\,\cos\theta\},\quad \mathbf{r}'=\{0,\,0,\,-a\cos\omega_0 t'\},\quad \boldsymbol{\beta}=\{0,\,0,\beta_0\sin\omega_0 t'\},\quad \beta_0\equiv\frac{a\omega_0}{c},$$

where a is particle's motion amplitude. (Note that a and ω_0 have to satisfy condition $\beta_0 \equiv a\omega_0/c \leq 1$.) Plugging this expression into Eq. (10.61), and performing vector multiplication, and introducing dimensionless integration variable $\xi \equiv \omega_0 t'$, we get

$$\frac{d\overline{\mathscr{P}_m}}{d\Omega}=\frac{Z_0q^2\beta_0^2m^2\omega_0^2\sin^2\theta}{8\pi^3}\left|I_m\right|^2,$$

where I_m is a dimensionless integral:

$$I_m\equiv\int_{-\pi}^{+\pi}\sin\xi\exp\{im(\xi-\alpha\cos\xi)\}d\xi,\quad \alpha\equiv\beta_0\cos\theta<1.$$

Transforming the integral as

$$I_m=\frac{1}{2i}\int_{-\pi}^{+\pi}\left(e^{i\xi}-e^{-i\xi}\right)e^{im\xi}e^{-im\alpha\cos\xi}d\xi=\frac{1}{2i}\int_{-\pi}^{+\pi}\left(e^{i(m+1)\xi}-e^{i(m-1)\xi}\right)e^{-im\alpha\cos\xi}d\xi,$$

we may notice that integrals of imaginary parts of $\exp\{i(m \pm 1)\xi\} = \cos(m \pm 1)\xi + i\sin(m \pm 1)\xi$, in our symmetric limits, vanish, so that

$$\begin{aligned}I_m&=\frac{1}{2i}\int_{-\pi}^{+\pi}\left[\cos(m+1)\xi-\cos(m-1)\xi\right]e^{-im\alpha\cos\xi}d\xi\\&=\frac{1}{i}\int_{0}^{+\pi}\cos(m+1)\xi e^{-im\alpha\cos\xi}d\xi-\frac{1}{i}\int_{0}^{+\pi}\cos(m-1)\xi e^{-im\alpha\cos\xi}d\xi,\end{aligned}$$

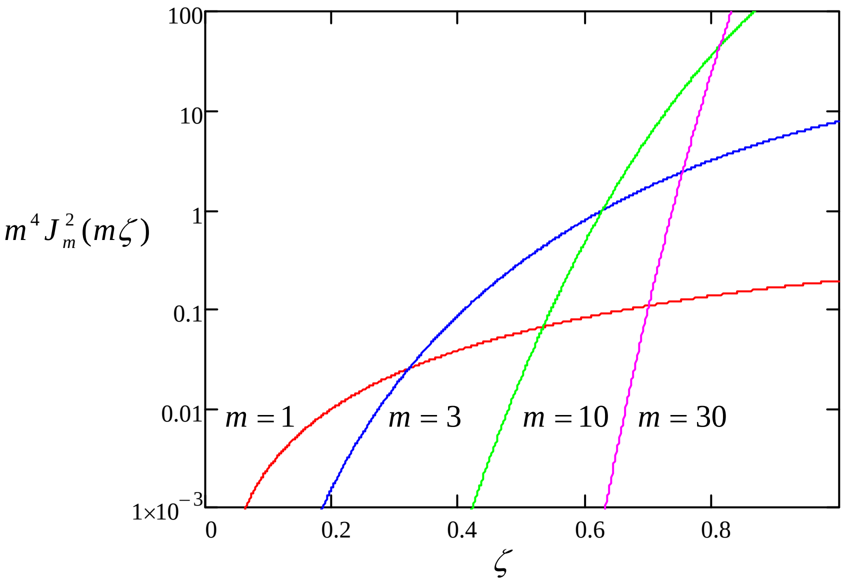
Apart from constant coefficients, these integrals are standard integral representations of the Bessel functions of the first kind,[2] so that we get

$$I_m = \frac{\pi}{i}\left[i^{m+1}J_{m+1}(m\alpha) - i^{m-1}J_{m-1}(m\alpha)\right] = i^m \pi\left[J_{m+1}(m\alpha) + J_{m-1}(m\alpha)\right] = i^m 2\pi\frac{m}{\alpha}J_m(m\alpha),$$

where for the last transition, the first of recurrent relations (2.143) has been used. To summarize, we have

$$\frac{d\overline{\mathscr{P}}_m}{d\Omega} = \frac{Z_0 q^2 \omega_0^2}{\pi}\tan^2\theta\, m^4 J_m^2(m\beta_0\cos\theta), \quad m \geq 1.$$

Figure below shows the dependence of the power of a few harmonics of radiation on the amplitude of particle's motion.



In the nonrelativistic limit ($\beta_0 \to 0$), only the first harmonic ($m = 1$) is nonvanishing, so that the radiation takes place only at the frequency of particle motion, and its angular distribution,

$$\frac{d\overline{\mathscr{P}}_1}{d\Omega} = \frac{Z_0 q^2 \omega_0^2}{\pi}\tan^2\theta\, J_1^2(\beta_0\cos\theta) \to \frac{Z_0 q^2 \omega_0^2}{\pi}\tan^2\theta\left(\frac{\beta_0\cos\theta}{2}\right)^2 = \frac{Z_0 q^2 \omega_0^2}{4\pi}\beta_0^2\sin^2\theta,$$

is fully described by the nonrelativistic formula (8.26). As the maximum value β_0 of the normalized velocity of the particle approaches 1, i.e. the particle becomes ultrarelativistic at least two small parts of its motion period, the intensity of high harmonics going in the direction of motion ($\cos\theta \approx 1$) grows rapidly, reflecting the narrow radiation cones generated at these parts – see Fig. 10.4 and its discussion.

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[2] See, e.g., MA Eq. (6.14).