Classical Electrodynamics II

Midterm exam with solutions

Problem M.1 (to be graded of 200 points). Find the lumped ac circuit equivalent to a TEM transmission line of length $l \sim \lambda$, with small cross-section area $A << \lambda^2$, as "seen" (measured) from one end, if the line's conductors are connected ("shortened") at the other end. Discuss the result's dependence on the signal frequency.

Solution: Neglecting losses in the transmission line, we can describe any "global" variable, e.g., voltage $V(z, t)$ or current $I(z, t)$, by a sum of two waves: one traveling to the shortened end, and another one reflected from it. Taking the shortened end's position for $z = 0$, for the complex amplitudes of these variables we can write (cf. Eq. (7.61) for plane waves):

$$V_\omega(z \le 0) = V_0\left(e^{ikz} - Re^{-ikz}\right), \quad I_\omega(z \le 0) = \frac{V_0}{Z_W}\left(e^{ikz} + Re^{-ikz}\right).$$

Now requiring the ratio V_ω/I_ω to vanish at the shorted end (at $z = 0$), we get $R = 1$, so that at the other end of the line (at $z = -l$) we get

$$Z(\omega) \equiv \frac{V_\omega}{I_\omega}\Big|_{z=-l} = Z_W \frac{e^{-ikl} - e^{ikl}}{e^{-ikl} + e^{ikl}} = -iZ_W \tan kl .$$

This result shows that in contrast to the line impedance $Z_W = (L_0/C_0)^{1/2}$, which does not depend on frequency, the effective ac impedance $Z(\omega)$ of the shorted line segment is a strong function of ω. In particular, at low frequencies $\omega << v/l$ (i.e. $kl << 1$) this expression reduces to

$$Z(\omega) \approx -iZ_W kl = -i\left(\frac{L_0}{C_0}\right)^{1/2} \omega\left(L_0 C_0\right)^{1/2} l = -i\omega L_0 l = -i\omega L \;,$$

where $L = L_0 l$ is the segment's inductance. This is a well-known expression for the complex impedance of a lumped inductance. (The negative sign is due to the fact that we are using the "physics" convention $\exp\{-i\omega t\}$ for the time dependence, while in electrical engineering the positive sign in more common.)

As kl is increased, $Z(\omega)$ grows faster than that of a lumped inductance, and diverges at $kl = \pi/2$, i.e. at wavelength $\lambda \equiv 2\pi/k = 4l$. This divergence means that in the absence of power losses, a finite voltage V_ω may be sustained by vanishing current. This resonant behavior is similar to that of an ac circuit consisting of a lumped inductance and lumped capacitance connected in parallel; it repeats at all frequencies where

$$k_n l = \frac{\pi}{2} + n\pi \;, \qquad (*)$$

i.e. $(n + 1/2)\lambda_n/2 = l$. These *parallel resonances* are interleaved, at

$$k'_n\, l = n\pi \;, \qquad (**)$$

by resonances of a different kind, at which $Z(\omega) = V_\omega/I_\omega$ vanishes, and hence a finite ac current may be sustained with little, if any, ac voltage. These are so-called *series resonances*, similar to those in series lumped *LC* circuits.

Problem M.2 (100 points). Calculate the contribution to resonator damping, due to small energy losses in the dielectric which fills it.

Solution: For the dispersion-free case, Eq. (7.184) of the lecture notes has a sinusoidal solution with frequency ω which satisfies characteristic equation

$$\omega^2 = k^2 v^2 \equiv \frac{k^2}{\varepsilon\mu},$$

where k is an eigenvalue of Eq. (7.187). Now acting in analogy with the derivation of Eq. (7.27), we can generalize this relation for a dispersive medium as

$$\omega^2 \equiv \frac{k^2}{\varepsilon(\omega)\mu(\omega)}. \qquad (*)$$

If the dielectric is lossy, then $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$, and $\mu(\omega) = \mu'(\omega) + i\mu''(\omega)$,[1] and the solution of the characteristic equation is complex, $\omega = \omega' + i\omega''$. This means that the time factor T(t) in Eq. (7.186) is not exactly sinusoidal, but decays with time:

$$\mathcal{T}(t) = \text{Re}\left[\mathcal{T}_\omega e^{-i\omega t}\right] = \text{Re}\left[\mathcal{T}_\omega e^{-i(\omega' + i\omega'')t}\right] = e^{\omega'' t}\,\text{Re}\left[\mathcal{T}_\omega e^{-i\omega' t}\right],$$

so that the resonator's average energy, proportional to T^2, decays as

$$\bar{\mathcal{E}}(t) = \bar{\mathcal{E}}(0)e^{2\omega'' t}.$$

Comparing this expression with Eq. (7.210), we see that the contribution of complex ω into the damping factor $\delta \equiv \omega/2Q$ is

$$\delta_{\text{dielectric}} = -\omega'' = -\text{Im}\,\omega.$$

If the losses are low, $\varepsilon''(\omega) << \varepsilon'(\omega)$, and $\mu''(\omega) << \mu(\omega)$, then $\omega'' << \omega'$, and the characteristic equation (*),

$$(\omega' + i\omega'')^2(\varepsilon' + i\varepsilon'')(\mu' + i\mu'') = k^2/\mu, \qquad (**)$$

may be solved by successive approximations. In the 0^{th} approximation, $\omega'' = 0$, while ω' is just the unperturbed, real eigenfrequency ω_0 of the resonator, which should be found self-consistently from equation $\omega_0 = k/[\varepsilon'(\omega_0)\mu'(\omega_0)]^{1/2}$. In the 1^{st} approximation, we can plug this result, $\omega' = \omega_0$, into Eq. (**) linearized in small ω'' and ε'':

$$2\omega'\omega''\varepsilon'\mu' + \omega'^2(\varepsilon'\mu'' + \varepsilon''\mu') \approx 0,$$

to get expression

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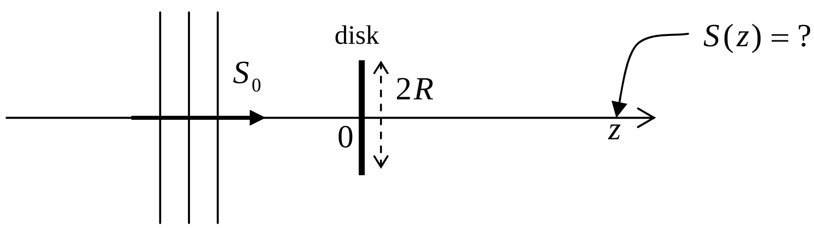
[1] For most dielectrics, $\mu(\omega)$ is very close to μ_0, and that $\mu'(\omega)$ is negligible.

$$\delta_{\text{dielectric}} = -\omega'' \approx \frac{\varepsilon'\mu'' + \varepsilon''\mu'}{2\omega'\varepsilon'\mu'} \approx \frac{\omega_0}{2}\left[\frac{\operatorname{Im}\varepsilon(\omega_0)}{\operatorname{Re}\varepsilon(\omega_0)} + \frac{\operatorname{Im}\mu(\omega_0)}{\operatorname{Re}\mu(\omega_0)}\right],$$

which may be rewritten for the reciprocal Q factor as

$$\left(\frac{1}{Q}\right)_{\text{dielectric}} \approx \frac{\operatorname{Im}\varepsilon(\omega_0)}{\operatorname{Re}\varepsilon(\omega_0)} + \frac{\operatorname{Im}\mu(\omega_0)}{\operatorname{Re}\mu(\omega_0)}.$$

Problem M.3 (250 points). A plane monochromatic wave is normally incident on an opaque disk of radius $R >> \lambda$. Use the Huygens principle to calculate the wave intensity at distance $z >> R$ behind the disk center (see Fig. below). Discuss the result.



Solution: According to Eq. (8.78) of the lecture notes, the complex amplitude of the wave in the observation point is

$$f_\omega(z) = f_0 \frac{k}{2\pi i}\int_R^\infty \rho' d\rho' \int_0^{2\pi} d\varphi' \frac{\exp\left\{ik\left(\rho'^2 + z^2\right)^{1/2}\right\}}{\left(\rho'^2 + z^2\right)^{1/2}}.$$

Using the axial symmetry of the problem, and Huygens principle conditions (8.70), $z >> R >> \lambda$, this expression may be simplified, just as has been done in Sec. 8.5 – see, e.g., Eq. (8.73):

$$f_\omega(z) \approx f_0 \frac{k}{iz} e^{ikz} \int_R^\infty \exp\left\{i\frac{k\rho'^2}{2z}\right\}\rho' d\rho' = f_0 \frac{k}{iz} e^{ikz} \frac{z}{k} \int_{kR^2/2z}^\infty \exp\{i\xi\} d\xi = -f_0 e^{ikz} \exp\{i\xi\}\Big|_{\frac{kz^2}{2R}}^{\infty}.$$

Since $\exp\{i\xi\} = \cos\xi + i\sin\xi$ is an oscillating rather than a decaying function, the result of the upper-limit substitution may not be apparent. However, since this is an analytical function, we can always add to ξ a small imaginary part $i\zeta$. As a result, the upper-limit substitution disappears, and now we can make the transition $\zeta \to 0$. As a result, we get

$$f_\omega(z) = f_0 e^{ikz} \exp\left\{i\frac{kR^2}{2z}\right\},$$

leading to a paradoxical answer

$$S = S_0.$$

Historically, this result was first obtained in 1818 by Poisson who used it as an argument *against* the theory of diffraction which had been developed by Fresnel (and against the wave theory of light as the whole). The following experiments have shown, however, that the pattern of diffraction on a disk with $R << z$ does indeed feature a light spot in the center.