PHY 505 **Classical Electrodynamics I** Fall 2010

Midterm Exam 1 with solutions

Problem M.1 (To be graded of 150 points). Calculate the spatial distributions of ϕ and **E,** created by a long, round cylinder of radius R, with the electric charge uniformly spread over its volume. Compare the result with that of for the uniformly charged sphere. Can you calculate the electrostatic energy of the system (per unit length)? If not, can you estimate the total energy of such cylinder of a finite length $L >> R$?

Solution: Due to the axial symmetry of the problem, we may write $\phi(\mathbf{r}) = \phi(\rho)$, $\mathbf{E}(\mathbf{r}) = \mathbf{n}_\rho E(\rho)$, where ρ is the distance from the cylinder's axis. Applying the Gauss law to a long cylinder of radius ρ, coaxial with the charged cylinder, we get

$$E(\rho) = \frac{\lambda}{2\pi\varepsilon_0} \times \begin{cases} \rho / R^2, & \text{for } \rho \leq R, \\ 1/\rho, & \text{for } \rho \geq R, \end{cases} \qquad (*)$$

where λ is the electric charge per unit length. Now integrating the relation $E(\rho) = -d\phi(\rho)/d\rho$ which follows, for our symmetry, from the general expression for gradient in cylindrical coordinates,[1] we get

$$\phi(\rho) = -\frac{\lambda}{2\pi\varepsilon_0} \times \begin{cases} \rho^2 / 2R^2 + c_1, & \text{for } \rho \leq R, \\ \ln(\rho / R) + c_2, & \text{for } \rho \geq R. \end{cases}$$

Since the electrostatic potential has to be continuous at $\rho = R$, the integration constants have to be related as

$$c_2 = c_1 + \frac{1}{2}.$$

Comparison of Eq. (*) with Eq. (1.19) and (1.22) of the lecture notes shows that while the field inside the cylinder changes similarly to that inside the uniformly charged sphere, outside the cylinder it changes much slower – as $1/\rho$ rather than $1/r^2$. Due the this change, integral (1.67) for the electrostatic energy diverges at large distances, even if calculated per unit length:

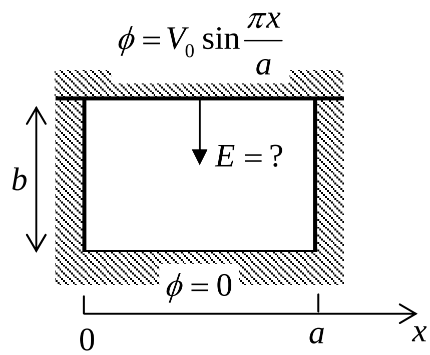
$$\frac{U}{L} = \frac{\varepsilon_0}{2} 2\pi \left[\int_0^R \left(\frac{\lambda}{2\pi\varepsilon_0} \frac{\rho}{R^2} \right)^2 \rho d\rho + \int_R^\infty \left(\frac{\lambda}{2\pi\varepsilon_0} \frac{1}{\rho} \right)^2 \rho d\rho \right] = \frac{\lambda^2}{4\pi\varepsilon_0} \left(\frac{1}{4} + \ln \rho \Big|_R^\infty \right).$$

However, this divergence is weak (logarithmic), indicating that a good estimate for the energy of a cylinder of finite length $L >> R$ may be obtained by just cutting the divergence at $\rho = L$, i.e. as

$$U \approx \frac{\lambda^2 L}{4\pi\varepsilon_0} \left(\frac{1}{4} + \ln \frac{L}{R} \right).$$

[1] See, e.g., MA Eq. (10.2).

Problem M.2 (300 points). Apply the variable separation method to find the electric field in the middle of the top lid of a long cylindrical box with the cross-section shown in Fig. on the right. The electrostatic potential of the bottom lid and side walls of the box is zero, while that of the top lid follows the sinusoidal function specified in the figure.



Solution: The separation of variables for this 2D problem is carried out just as for the 3D box discussed in class, giving

$$\phi(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{\pi n x}{a} \sinh\frac{\pi n y}{a}.$$

Comparing this solution, taken on the top lid ($y = b$), with the fixed potential of the lid, we see that only one coefficient c_n, with $n = 1$, survives:

$$V_0 = c_1 \sinh\frac{\pi b}{a},$$

so that the final distribution of the potential inside the box is

$$\phi(x,y) = V_0 \sin\frac{\pi x}{a} \frac{\sinh\frac{\pi y}{a}}{\sinh\frac{\pi b}{a}}.$$

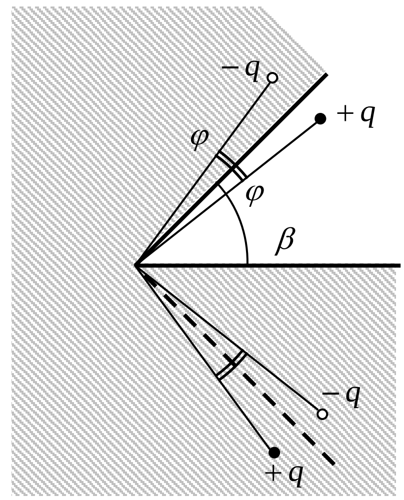
From this solution, the electric field in the middle of the top lid is

$$E(\frac{a}{2}, b) = \left|\frac{\partial\phi}{\partial y}\right|_{\substack{x=a/2,\\ y=b}} = V_0 \frac{\pi}{a} \operatorname{cotanh}\frac{\pi b}{a}.$$

If the box is tall and/or narrow ($a << b$), then cotanh($\pi b/a$) ≈ 1, and the electric field does not depend on b: $E \approx \pi V_0/a$. Such independence is due to the fact that all field lines starting from the top lid end up at the side walls and do not reach the bottom lid, so its location is irrelevant. In the opposite limit ($b << a$), the field is independent of a: $E \approx V_0/b$. This is natural, because in this case the system is essentially a plane capacitor, so that side wall location does not affect the field in its middle.

Problem M.3 (200 points). For what values of the corner angle β, the 2D boundary problem, shown in Fig. on the right, can be solved using a finite number of image charges? (Justify your answer.)

Solution: Let the angle between the direction to the charge and the nearest wall equal φ (see Fig.), so that its angular distance from the opposite wall is (β - φ). In order to satisfy the boundary condition (ϕ = 0) at the nearest wall, let us add the charge's reflection in the nearest wall: a negative image charge at the same distance from the corner, and angle ($\beta + \varphi$).



Still, the boundary condition at the opposite wall is not satisfied. In an attempt to correct that, let us reflect the pair we have in that wall, i.e. add charge $+q$ at angle ($-\beta - \varphi$) and charge $-q$ at ($-\beta + \varphi$) – see Fig. We see that the resulting two pairs of charges have the same angular direction of their dipole moments, and their centers are separated by angle 2β.

Now, the boundary condition on the first wall (nearest to the real charge) is not satisfied again. In order to correct that, let us repeat the reflection procedure again, now in that wall. At this new reflection, the first pair does not change, but the second pair gives a new (third) one, with the same angular direction of the dipole moment. Now the pairs occupy angle 3β, but (for in the general case), the boundary conditions on the second wall are not satisfied again.

Repeating this charge pair reflection procedure again and again (say, n times after the formation of the first charge pair), we get n such pairs, with centers within a sector $2n\beta$ wide. In order to have the process stopped (i.e. the boundary conditions satisfied on both walls), we need the last generated pair to overlap with an already existing pair, so that $2n\beta$ should equal 2π. This is possible if $\beta = \pi/n$, with $n = 1, 2, 3, \ldots$

Superficially, the set of possible β is broader, and is given by expression $(m/n)\pi$, with m also an arbitrary integer. Indeed, after n pair reflections the pairs would cover angle $2n\beta = 2m\pi$, i.e. an integer number of 2π, and start overlapping with old ones. However, at $m > 1$ at least one of the image charges will be in the free-space (observable) region, so that the field in that region would have more than one point-charge singularity, and thus would not satisfy our initial problem.