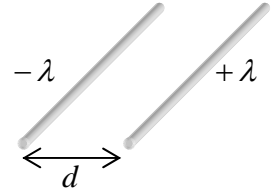


Problem O.1. Two thin, straight parallel filaments, separated by distance d , carry equal and opposite uniformly distributed charges with linear density λ - see Fig. on the right. Find the electrostatic force (per unit length) of the Coulomb interaction between the wires. Compare the result with the Coulomb law for the force between the point charges, and interpret their difference.



Solution: Using result (*) of the previous problem, we get

$$\frac{F}{L} = \frac{qE(d)}{L} = \lambda E(d) = \frac{\lambda^2}{2\pi\epsilon_0 d}.$$

Note that the force drops with distance as $1/d$, rather than as $1/r^2$ for point charges. Such different scaling of interaction for systems of different dimensionality is very typical for physics at large.

Problem O.2. Can one create the electrostatic fields presented below by sets of their components in Cartesian coordinates $\{x, y, z\}$, in a finite region of space?

- (i) $\{yz, xz, xy\}$
- (ii) $\{xy, xy, yz\}$

Solution: Let us calculate curl of both supposed fields, using the definition of that operator - see, e.g., MA Eq. (8.5):

$$\nabla \times \mathbf{E} = \left\{ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right\}.$$

For field (i) we get $\nabla \times \mathbf{E} = \{x - x, y - y, z - z\} \equiv 0$, while for field (ii), $\nabla \times \mathbf{E} = \{z - 0, 0 - 0, y - x\}$ vanishes only in one point ($x = y = z = 0$). However, according to Eq. (1.28) of the lecture notes, curl of real electric field has to equal zero at any point where it exists; hence field (i) can be created in a region of finite size, while field (ii) cannot. (The fact that field (i) has zero divergence, i.e. requires $\rho(\mathbf{r}) \equiv 0$ within the region of its existence, does not prevent it from being realistic, because it may be created by electric charges outside of that particular region.)

Problem O.3. Calculate the force (per unit area) exerted on a conducting surface by an external electric field. Compare the result with the definition of the electric field given by Eq. (1.6) of the lecture notes, and comment.

Solution: The (crucial!) qualifier “external” means that the field $E = E_n$ at the surface does not change at its small virtual displacement $\delta \mathbf{r}$. $\delta x \equiv \mathbf{n} \cdot \delta \mathbf{r}$. Hence the change of field’s energy (1.67) per unit area is just

$$\frac{\delta U}{A} = \frac{1}{A} \delta \left(\frac{\epsilon_0}{2} \int E^2 d^3 r \right) = \frac{\epsilon_0}{2} E^2 \frac{1}{A} \delta V = \frac{\epsilon_0}{2} E^2 (-\delta x),$$

where V is the volume outside the conductor (filled with the field), and $\delta x \equiv \mathbf{n} \cdot \delta \mathbf{r}$ is the conductor surface's displacement toward the field-occupied space. Hence the force is directed out of the conductor, and equal to

$$\frac{F}{A} = -\frac{1}{A} \frac{\delta U}{\delta x} = \frac{\epsilon_0}{2} E^2.$$

From the point of view of the electric field definition (1.6), coefficient $1/2$ in this expression may be interpreted as follows. The applied field equals to E on one side of the induced surface charge layer, with areal density $\sigma = \epsilon_0 E$, but vanishes on its another side, so that the average field inside the layer is $E_{\text{ave}} = E/2$. Hence the force per unit area may be calculated as $F/A = \sigma E_{\text{ave}} = (\epsilon_0 E)(E/2)$, in agreement with the above result.

Problem O.4. Read (and understand :-)) the discussion of the Schwarz-Christoffel integral in Sec. 2.4 of the lecture notes, and its application to the problem shown in Fig. 2.10. Complete the problem's solution by calculating the distribution of the surface charge of the conducting semi-planes. Can you calculate the mutual capacitance of the planes (per unit length)?

Solution: Near the electrode surface ($y \rightarrow 0$, $x^2 > t^2$), Eq. (2.83) of the lecture notes yields

$$\phi \rightarrow \frac{V}{\pi} \arcsin \left\{ 1 - \frac{y^2}{4x} \left[\frac{1}{x+t} + \frac{1}{x-t} \right] \right\} = \frac{V}{\pi} \arcsin \left\{ 1 - \frac{y^2}{2(x^2 - t^2)} \right\}.$$

From here, the only non-vanishing (vertical) component of the field is

$$E_n \Big|_{y=\pm 0} = \mp \frac{\partial \phi}{\partial y} \Big|_{y=\pm 0} = \frac{V}{\pi(x^2 - t^2)^{1/2}},$$

so that the double-side charge density is

$$\sigma = \frac{2\epsilon_0 V}{\pi(x^2 - t^2)^{1/2}}.$$

The divergence at $x \rightarrow t$ is not too strong (integrable), just as the thin-disk problem solved in Sec. 2.4, but in contrast to that problem, the total charge of each semi-plane (per unit length) is still infinite, because the corresponding integral,

$$\frac{Q}{L} = \int_t^\infty \sigma dx = \frac{2\epsilon_0 V}{\pi} \int_t^\infty \frac{dx}{(x^2 - t^2)^{1/2}} = \frac{2\epsilon_0 V}{\pi} \ln \left[x + (x^2 - t^2)^{1/2} \right]_{x=t}^{x \rightarrow \infty}, \quad (*)$$

diverges at the upper limit, albeit very slowly (logarithmically). Hence in order to calculate the mutual capacitance per unit length, $C_m/L = Q/LV$, one need to make one more step from this idealized model to reality, for example, take into account a finite width w of each semi-plate. Even not solving the resulting problem exactly, we can use Eq. (*) to make a semi-quantitative prediction of the result for the case $w \gg t$:

$$\frac{C_m}{L} = \frac{Q}{LV} \approx \frac{2\epsilon_0}{\pi} \ln \left[x + (x^2 - t^2)^{1/2} \right]_{x=t}^{x=w \gg t} \approx \frac{2\epsilon_0}{\pi} \ln \frac{2w}{t}.$$

Problem O.5. Calculate the potential energy of a point charge placed in the center of a spherical cavity which was cut inside a grounded conductor.

Solution: First, let us calculate force \mathbf{F} exerted by the conductor on a point charge q' shifted by distance $d' < R$ from cavity's center. As follows from the discussion of the complementary problem in Sec. 2.6 of the lecture notes (Fig. 2.27), both the Poisson equation inside the cavity and the boundary conditions at the conductor's surface may be satisfied by adding, to the Coulomb field of the real charge, that of an image charge q placed at distance $d > R$ from the center, with q and d related to q' and d' by Eqs. (2.198). Solving these equations for q and d (now functions of given q' and d'), we get

$$d = \frac{R^2}{d'}, \quad q = -q' \frac{d}{R} = -q' \frac{R}{d'}.$$

From here, the magnitude of force \mathbf{F} (directed toward the image charge, i.e. the nearest point of the conducting surface) is

$$F(d') = \frac{|qq'|}{4\pi\epsilon_0} \frac{1}{(d-d')^2} = \frac{q'^2}{4\pi\epsilon_0} \frac{R}{d'} \frac{1}{(R^2/d' - d')^2} = \frac{q'^2}{4\pi\epsilon_0} \frac{Rd'}{(R^2 - d'^2)^2}.$$

As a sanity check, the force vanishes at $d' \rightarrow 0$ (in agreement with the problem's symmetry), while at $d' \rightarrow R$ it tends to the expression,

$$F_{\text{plane}}(R - \varepsilon) = \frac{q'^2}{4\pi\epsilon_0} \frac{1}{(2\varepsilon)^2}, \quad \varepsilon \equiv R - d',$$

which we would get (e.g., from Eq. (2.191) of the lecture notes) for the charge at distance ε from a conducting plane.

Now the required potential energy might be calculated as the (minus) integral of the force on a path from infinity to the cavity center. Our results are only valid inside the sphere, but we may circumvent this problem by extending the integration from the center to distance $\varepsilon \ll R$ to the surface, and subtracting from the result the potential energy (2.191) provided by at that point by a conducting plane (explain why):

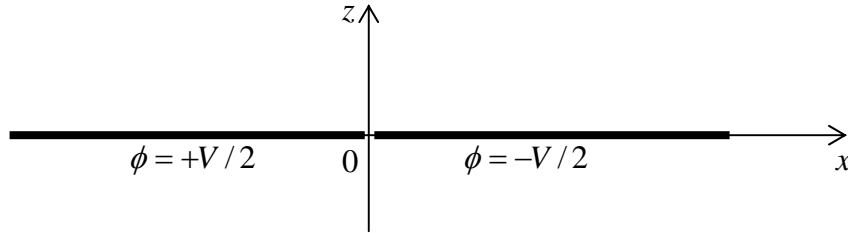
$$U(0) = - \int_0^{R-\varepsilon} F(x) dx - U_{\text{plane}}(R - \varepsilon) = - \frac{q'^2}{4\pi\epsilon_0} \left[\int_0^{R-\varepsilon} \frac{R^2 x}{(R^2 - x)^2} dx - \frac{1}{4\varepsilon} \right] = - \frac{q'^2}{8\pi\epsilon_0 R}.$$

This result is (by magnitude) twice larger than what a conducting plane would give at the same distance from it:

$$U_{\text{plane}}(0) = - \frac{q'^2}{16\pi\epsilon_0 R}.$$

The reason for such increase is clear: for a spherical cavity, all induced surface charges are at the same distance (R) from the initial charge, while only the closest charges on the conducting plane are at such distance, so that the effect of their attraction is smaller.

Problem O.6. A conducting plane located at $z = 0$ is separated into two parts with a very narrow, straight cut along axis y , and voltage V is fixed between the resulting half-planes, as shown in Fig. below. Use the Green's function method to find the distribution of the electrostatic potential in all the space, and the electric field on the symmetry plane ($x = 0$). Looking at the result, could be the problem solved in a simpler way?



Solution: For our cylindrical geometry ($\partial/\partial y = 0$), and in the absence of free charges, the basic formula of the Green's function theory is reduced to

$$\phi(\rho) = \frac{1}{4\pi} \sum_k \phi_k \oint_{L_k} \frac{\partial G(\mathbf{\rho}, \mathbf{\rho}')}{\partial n'} dr',$$

where, for our choice of coordinates, $\mathbf{\rho} = \{x, z\}$, while $G(\mathbf{\rho}, \mathbf{\rho}')$ is the 2D Green's function, and the integration should be extended along all boundaries of conductors' cross-sections (numbered with index k). For a semi-space limited by a conducting plane, the Green's function and its normal derivative have been calculated in Homework Problem 5.2:

$$G(\mathbf{\rho}, \mathbf{\rho}') = -2 \ln|\mathbf{\rho} - \mathbf{\rho}'| + 2 \ln|\mathbf{\rho} - \mathbf{\rho}''| = -\ln[(x-x')^2 + (z-z')^2] + \ln[(x-x')^2 + (z+z')^2], \quad (*)$$

$$\left. \frac{\partial G}{\partial z'} \right|_{z'=0} = \frac{4z}{(x-x')^2 + z^2},$$

so that for our current problem we get

$$\phi = \frac{(+V/2)}{4\pi} \int_{-\infty}^0 \frac{4z}{(x-x')^2 + z^2} dx' + \frac{(-V/2)}{4\pi} \int_0^{+\infty} \frac{4z}{(x-x')^2 + z^2} dx' = -\frac{V}{\pi} \arctan \frac{x}{|z|}.$$

On the symmetry plane $x = 0$, $\phi = 0$, so that the electric field does not have a vertical component. Its only, horizontal component

$$E_x = -\left. \frac{\partial \phi}{\partial x} \right|_{x=0} = \frac{V}{\pi|z|}.$$

Just as one could expect, the field increases (indeed, diverges) as we approach the infinitely narrow gap which holds a finite voltage drop V .

Alternatively, the problem could be solved just by noticing that in polar coordinates, both the Laplace equation and our particular boundary conditions are satisfied by a simple linear function of angle, $\phi(\varphi) = V(|\varphi|/\pi - 1/2)$, independent of ρ .