Optional Problems Set 2 with solutions

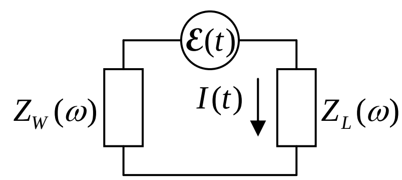
Problem O.8. What lumped ac circuit is equivalent to the system shown in Fig. 7.19b of the lecture notes, with monochromatic incident wave of power P_i? Assume that the wave reflected from the load circuit does not return to it.

Solution: It is intuitively clear that the circuit has to include:

(i) the lumped load with impedance $Z_L(\omega)$,

(ii) a passive lumped element with impedance Z_W, presenting passive properties of the transmission line, and

(iii) some signal generator, presenting the incident wave.



Following these arguments, let us try to use the circuit shown on the right, where $\mathcal{E}(t)$ is an e.m.f. presenting the incident wave. An elementary calculation of the complex amplitude of the current in the load and voltage drop across it yields $I_\omega = \mathcal{E}_\omega/[Z_L(\omega) + Z_W(\omega)]$, $V_\omega = I_\omega Z_L(\omega) = \mathcal{E}_\omega Z_L(\omega)/[Z_L(\omega) + Z_W(\omega)]$, so that the calculation similar to the derivation of Eq. (7.40) yields the following expression for the average power absorbed in the load:

$$\overline{\mathscr{P}}_L = \frac{1}{2} V_\omega I_\omega^* = \frac{1}{2} \frac{|\mathcal{E}_\omega|^2}{|Z_L(\omega) + Z_W(\omega)|^2} \operatorname{Re} Z_L(\omega) . \qquad (*)$$

On the other hand, we can calculate this power in the genuine circuit (Fig. 7.20) by subtracting from the incident power P_i the reflected wave power $P_R = (1/2)P_i|R|^2$, where R is given by Eq. (7.115):

$$\overline{\mathscr{P}}_L = \mathscr{P}_i\left(1 - |R|^2\right) = \mathscr{P}_i\left(1 - \frac{|Z_L(\omega) - Z_W(\omega)|^2}{|Z_L(\omega) + Z_W(\omega)|^2}\right) = \mathscr{P}_i \frac{|Z_L(\omega) + Z_W(\omega)|^2 - |Z_L(\omega) - Z_W(\omega)|^2}{|Z_L(\omega) + Z_W(\omega)|^2} .$$

In the most important case of a loss-free line, its impedance Z_W is real, and this expression is reduced to

$$\overline{\mathscr{P}}_L = \mathscr{P}_i \frac{4 Z_W(\omega)}{|Z_L(\omega) + Z_W(\omega)|^2} \operatorname{Re} Z_L(\omega) . \qquad (**)$$

Comparing Eqs. (*) and (**), we see that they give similar results (and hence the lumped circuit is equivalent to the wave system shown in Fig. 20), if the effective e.m.f. has amplitude

$$|\mathcal{E}_\omega| = [8\mathscr{P}_i Z_W(\omega)]^{1/2} .$$

Such presentation of a transmission line by its impedance and e.m.f. is valid even if the lumped load is nonlinear and/or time-dependent, and is broadly used for the analysis of not only linear, but also nonlinear and parametric microwave devices.[1]

[1] For a brief discussion of the nonlinear and parametric interactions, see, e.g., CM Sec. 5.5.

Problem O.9. Calculate the skin-effect contribution to the attenuation coefficient α of

(i) the basic (H_{11}) mode, and

(ii) the H_{01} mode

of waves in a metallic waveguide with the circular cross-section (Fig. 7.22a of the lecture notes), and analyze the low-frequency ($\omega \to \omega_c$) and high-frequency ($\omega >> \omega_c$) behavior of α for these two modes.

Solutions:

(i) For the H_{11} mode, the longitudinal component H_z of the magnetic field (or rather its complex amplitude) is described by Eq. (7.140) of the lecture notes:

$$H_z(\rho,\varphi) = H_l J_1\left(\xi'_{11}\frac{\rho}{R}\right)e^{i\varphi}, \quad \text{with } \xi'_{11} \approx 1.84.$$

Plugging this result into Eqs. (7.117) with $E_z = 0$, and using the general expressions[2]

$$(\nabla_t f)_\rho = \frac{\partial f}{\partial \rho}, \quad (\nabla_t f)_\varphi = \frac{1}{\rho}\frac{\partial f}{\partial \varphi},$$

for transversal components of the electric and magnetic fields we get

$$E_\rho(\rho,\varphi) = -\frac{k}{k_t^2} ZH_l \frac{1}{\rho} J_1\left(\xi'_{11}\frac{\rho}{R}\right)e^{i\varphi}, \quad E_\varphi(\rho,\varphi) = i\frac{k}{k_t} ZH_l J'_1\left(\xi'_{11}\frac{\rho}{R}\right)e^{i\varphi},$$

$$H_\rho(\rho,\varphi) = -i\frac{k_z}{k_t} H_l J'_1\left(\xi'_{11}\frac{\rho}{R}\right)e^{i\varphi}, \quad H_\varphi(\rho,\varphi) = -\frac{k_z}{k_t^2} H_l \frac{1}{\rho} J_1\left(\xi'_{11}\frac{\rho}{R}\right)e^{i\varphi},$$

where $k = \omega(\varepsilon\mu)^{1/2}$, $Z = (\mu/\varepsilon)^{1/2}$, $k_t = \xi'_{11}/R$, and k_z should be found from the general Eq. (7.98b):

$$k_z = \left(k^2 - k_t^2\right)^{1/2}.$$

From here, the average areal density of power loss due to the skin effect, given by Eq. (7.206), is

$$\frac{\overline{\mathscr{P}}_{\text{loss}}}{A} = \frac{\mu\omega\delta(\omega)}{4}\left[\left|H_\varphi(R,\varphi)\right|^2 + \left|H_z(R,\varphi)\right|^2\right] = \frac{\mu\omega\delta(\omega)}{4}\left|H_l\right|^2 J_1^2\left(\xi'_{11}\right)\left[\frac{k_z^2}{k_t^4 R^2} + 1\right].$$

Integrating this expression over the cross-section's perimeter,[3] we find the average power loss per unit length of the waveguide:

$$\frac{d\overline{\mathscr{P}}_{\text{loss}}}{dz} = 2\pi R\frac{\overline{\mathscr{P}}_{\text{loss}}}{A} = \frac{\pi}{2} J_1^2\left(\xi'_{11}\right)\mu\omega\delta(\omega) R\left|H_l\right|^2\left[\frac{1}{(k_t R)^2}\frac{k_z^2}{k_t^2} + 1\right].$$

By definition of ξ'_{11}, $J_1(\xi'_{11})$ is just the first maximum of Bessel function $J_1(\xi'_{11})$, close to 0.58 (see, e.g., Fig. 2.16), while $(k_t R)^2 = (\xi'_{11})^2 \approx 3.39$.

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[2] See, e.g., MA Eq. (10.2).

[3] The fact that the power loss (as well as the Poynting vector below) does not depend on φ should not be surprising, because (as was noted in Sec. 7.7 of the lecture notes), our expressions actually describe circularly polarized waves. Since a linearly-polarized wave may be presented as a linear superposition of two independent circularly-polarized ones, our final result (the attenuation constant) is valid for such a wave as well.

The average longitudinal component of the Poynting vector is

$$\overline{S_z} = \frac{E_\rho H_\varphi^* - E_\varphi H_\rho^*}{2} = \frac{Z}{2}\left|H_l^2\right|\frac{kk_z}{k_t^4}\left[\frac{1}{\rho^2} J_1^2\left(\xi_{11}'\frac{\rho}{R}\right) + \frac{\xi_{11}'^2}{R^2} J_1'^2\left(\xi_{11}'\frac{\rho}{R}\right)\right].$$

Integrating this expression over waveguide's cross-section for the average propagating power we get

$$\overline{\mathscr{P}} = \int_0^{2\pi} d\varphi \int_0^R \rho\, d\rho\, \overline{S}_z = \pi Z\left|H_l^2\right|\frac{kk_z}{k_t^2}\int_0^R\left[\frac{1}{k_t^2\rho^2} J_1^2\left(\xi_{11}'\frac{\rho}{R}\right) + J_1'^2\left(\xi_{11}'\frac{\rho}{R}\right)\right]\rho\, d\rho = \frac{\pi I}{\xi_{11}'^2} Z\left|H_l^2\right| kk_z R^4,$$

where I is a dimensionless constant:

$$I \equiv \int_0^{\xi_{11}'}\left[\frac{J_1^2(\xi)}{\xi^2} + J_1'^2(\xi)\right]\xi\, d\xi = \frac{1}{4}\int_0^{\xi_{11}'}\left[J_0^2(\xi) + J_2^2(\xi)\right]\xi\, d\xi \approx 0.291\,.$$

From here, the attenuation constant is

$$\alpha \equiv \frac{1}{\overline{\mathscr{P}}}\frac{d\overline{\mathscr{P}}_{\text{loss}}}{dz} = \frac{J_1^2(\xi_{11}')}{2I}\frac{\delta(\omega)}{k_z R^3}\left[\frac{k_z^2}{k_t^2} + \xi_{11}'^2\right].$$

The attenuation behavior is very similar to that for the rectangular waveguide: α diverges when the wave frequency ω is reduced to the cutoff value $\omega_c = k_t/(\varepsilon\mu)^{1/2}$, because at that point $k_z \to 0$, and increases as $\delta(\omega)k \propto \omega^{1/2}$ at high frequencies $\omega >> \omega_c$ where $k_z \approx k >> k_t$. (The minimum of α is reached between these two extremes, at $\omega \approx 3\ \omega_c$.)

(ii) For the H_{01} mode, with the longitudinal field described by Eq. (7.137) with $n = 0$ and $m = 1$,

$$H_z(\rho,\varphi) = H_l J_0\left(\xi_{01}'\frac{\rho}{R}\right), \quad \xi_{01}' \approx 3.83\,,$$

i.e. independent of the azimuthal angle, the absolutely similar calculations give even simpler results:

$$E_\rho(\rho,\varphi) = 0, \quad E_\varphi(\rho,\varphi) = -i\frac{k}{k_t^2} ZH_l \frac{\xi_{01}'}{R} J_0'\left(\xi_{01}'\frac{\rho}{R}\right) = i\frac{k}{k_t^2} ZH_l \frac{\xi_{01}'}{R} J_1\left(\xi_{01}'\frac{\rho}{R}\right),$$

$$H_\rho(\rho,\varphi) = i\frac{k_z}{k_t^2} H_l \frac{\xi_{01}'}{R} J_0'\left(\xi_{01}'\frac{\rho}{R}\right) = -i\frac{k_z}{k_t^2} H_l \frac{\xi_{01}'}{R} J_1\left(\xi_{01}'\frac{\rho}{R}\right), \quad H_\varphi(\rho,\varphi) = 0\,.$$

As a result, the skin-effect losses are determined only by the longitudinal component of **H**:

$$\frac{\overline{\mathscr{P}}_{\text{loss}}}{A} = \frac{\mu\omega\delta(\omega)}{4}\left|H_z(R,\varphi)\right|^2 = \frac{\mu\omega\delta(\omega)}{4}\left|H_l\right|^2 J_0^2(\xi_{01}'),$$

$$\frac{d\overline{\mathscr{P}}_{\text{loss}}}{dz} = R\int_0^{2\pi}\frac{\overline{\mathscr{P}}_{\text{loss}}}{A} d\varphi = \frac{\pi}{2} J_0^2(\xi_{01}')\mu\omega\delta(\omega)R\left|H_l\right|^2.$$

The longitudinal component of the Poynting vector is also contributed by only one product:

$$\overline{S_z} = -\frac{E_\varphi H_\rho^*}{2} = \frac{Z}{2}\left|H_l^2\right|\frac{kk_z}{k_t^4 R^2} J_1^2\left(\xi'_{01}\frac{\rho}{R}\right),$$

$$\overline{\mathscr{P}} = \int_0^{2\pi} d\varphi \int_0^R \rho\, d\rho\, \overline{S}_z = \pi Z\left|H_l^2\right|\frac{kk_z}{k_t^4}\int_0^R J_1^2\left(\xi'_{01}\frac{\rho}{R}\right)\rho\, d\rho = \pi\, I' Z\left|H_l^2\right|\frac{kk_z}{k_t^4}R^2, \quad I' \equiv \int_0^{\xi'_{01}} J_1^2(\xi)\xi\, d\xi \approx 0.632$$

As a result, the expression for the attenuation constant,

$$\alpha \equiv \frac{1}{\overline{\mathscr{P}}}\frac{\overline{d\mathscr{P}}_{\text{loss}}}{dz} = \frac{J_0^2\left(\xi'_{01}\right)}{2I'}\frac{\delta(\omega)k_t^4}{k_z R},$$

has a different frequency dependence: while still diverging at $\omega \to \omega_c$ (where $k_z \to 0$), it is proportional to $\delta(\omega)/k \propto \omega^{-1/2}$ at $\omega/\omega_c \to \infty$ where $k_z \to k >> k_t$.

This property, which makes H_{01} (as well as higher H_{0m} modes) so attractive for long-distance microwave energy transfer, is due to the absence of the azimuthal component of the magnetic field and the radial component of the electric field. At the wall surface ($\rho = R$), where function $J_1(\xi'_{01}\rho/R) \equiv -J'_0(\xi'_{01}\rho/R)$ vanishes, two other transversal components of the fields also vanish. As a result, the skin-effect losses are due only to the longitudinal component of the magnetic field, which decreases (at fixed wave power) as wave's frequency is increased and its structure becomes closer to TEM.

Problem O.10. Use the Born approximation to calculate the differential cross-section of plane wave scattering on a right, circular cylinder of length L and radius ρ, oriented along wave's electric field. Formulate the limitations of your result.

Solution: In order to use the key Eq. (8.62) of the lecture notes, we have to calculate the phase integral (8.63):

$$I(\mathbf{q}) \equiv \int_V \exp\{-i\mathbf{q}\cdot\mathbf{r}'\}d^3r' = \int_{-L/2}^{+L/2} dz' e^{-iq_z z'} \int_0^\rho \rho' d\rho' \int_0^{2\pi} d\varphi\, e^{-iq_\perp \rho' \cos\varphi},$$

where q_z is the component of the scattering vector $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$ along cylinder's axis (taken for coordinate axis z), $\mathbf{q}_\perp$ is its component in the plane of cylinder's cross-section, and φ is the angle between that component and vector $\boldsymbol{\rho}' \equiv \mathbf{r}' - \mathbf{n}_z z'$. The first integral is simple (see Eqs.(8.64)-(8.66) and Fig. 8.7),

$$\int_{-L/2}^{+L/2} e^{-iq_z z'} dz' = L\,\text{sinc}\frac{q_z L}{2},$$

while the second one may be expressed via a Bessel function of the first kind, using their integral representation[4] and the recurrent relation (2.143b) with $n = 1$:

$$\int_0^\rho \rho' d\rho' \int_0^{2\pi} d\varphi\, e^{-iq_\perp \rho' \cos\varphi} = 2\pi \int_0^\rho J_0(q_\perp \rho')\rho' d\rho' = \frac{2\pi}{q_\perp^2}\int_0^{q_\perp \rho} J_0(\xi)\xi d\xi = \frac{2\pi}{q_\perp^2}\left[\xi J_1(\xi)\right]_0^{q_\perp \rho} = \frac{2\pi\rho}{q_\perp} J_1(q_\perp \rho).$$

With these substitutions, Eqs. (8.62)-(8.63) yield

[4] See, e.g., MA Eq. (6.14a) with $n = 0$.

$$\frac{d\sigma}{d\Omega}=\frac{k^4}{(4\pi)^2}(\varepsilon_r-1)^2\left|I(\mathbf{q})\right|^2\sin^2\theta=\frac{k^4L^2\rho^4}{16}(\varepsilon_r-1)^2\sin^2\theta\left(\operatorname{sinc}\frac{q_zL}{2}\right)^2\left(\frac{J_1(q_\perp\rho)}{(q_\perp\rho/2)}\right)^2,$$

where the factors in the last parentheses have been grouped to emphasize that this function is very much similar to sinc, and in particular tends to 1 when its argument (in our case, $q_\perp\rho$) tends to zero. Due to this grouping, it is evident that for a very small cylinder, $q^3V << 1$, where $V \equiv \pi\rho^2L$, our result reduces to Eq. (8.53).

Problem O.11. Show that two successive Lorentz transforms in the same direction, with velocities *u'* and *v*, are equivalent to a single transform with velocity *u* given by Eq. (9.25) of the lecture notes.

Solution: Let us denote normalized velocity u'/c as β_1, and v/c as β_2, with corresponding indices of γ. Then for the first transform, Eq. (9.19b) of the lecture notes gives

$$x_1=\gamma_1(x-\beta_1ct),\quad ct_1=\gamma_1(ct-\beta_1x),$$

and the second transform we get

$$\begin{aligned}x_2&=\gamma_2(x_1-\beta_2ct_1)=\gamma_1\gamma_2[(x-\beta_1ct)-\beta_2(ct-\beta_1x)]=\gamma_1\gamma_2[(1+\beta_1\beta_2)x-(\beta_1+\beta_2)ct],\\ ct_2&=\gamma_2(ct_1-\beta_2x_1)=\gamma_1\gamma_2[(ct-\beta_1x)-\beta_2(x-\beta_1ct)]=\gamma_1\gamma_2[(1+\beta_1\beta_2)ct-(\beta_1+\beta_2)x].\end{aligned}\qquad(*)$$

Let us spell out the product $\gamma_1\gamma_2$ involved in this result:

$$\gamma_1\gamma_2=\frac{1}{(1-\beta_1^2)^{1/2}}\frac{1}{(1-\beta_2^2)^{1/2}}=\frac{1}{[(1-\beta_1^2)(1-\beta_2^2)]^{1/2}}\frac{1}{(1+\beta_1^2\beta_2^2-\beta_1^2-\beta_2^2)^{1/2}}$$

On the other hand, in our new notation, Eq. (9.25) takes the form

$$\beta=\frac{\beta_1+\beta_2}{1+\beta_1\beta_2},.$$

so that the corresponding Lorentz factor is[5]

$$\gamma=\frac{1}{(1-\beta^2)^{1/2}}=\frac{1}{\left[1-\dfrac{(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}\right]^{1/2}}=\frac{1+\beta_1\beta_2}{(1+\beta_1^2\beta_2^2-\beta_1^2-\beta_2^2)^{1/2}}=\gamma_1\gamma_2(1+\beta_1\beta_2),$$

As a result, the direct transform from $\{x, t\}$ to $\{x_2, t_2\}$, using these β and γ, is

$$x_2=\gamma(x-\beta ct)=\gamma_1\gamma_2(1+\beta_1\beta_2)\left(x-\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}ct\right),$$

$$ct_1=\gamma(ct-\beta x)=\gamma_1\gamma_2(1+\beta_1\beta_2)\left(ct-\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}x\right),$$

i.e. evidently the same results as Eqs. (*) following from the sequential transform.

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[5] Note that γ is very much different from $\gamma_1\gamma_2$, so that the length contraction and time dilation formulas (9.20) and (9.21) cannot be applied sequentially! (Explain why.)

Problem O.12. A particle with rest mass m decays into 2 particles with rest masses m_1 and m_2. Prove that the total energy of the first decay product, in the rest frame of the decayed particle, is

$$\mathcal{E}_1 = \frac{m^2 + m_1^2 - m_2^2}{2m} c^2 .$$

Solution: In the rest frame of the decayed particle, its momentum p was zero, and energy was mc^2. Hence, due to the energy and momentum conservation at the decay, the 4-momenta of the decay products may be presented as

$$p_1^\alpha = \left\{ \frac{\mathcal{E}_1}{c}, \mathbf{p}_1 \right\}, \quad p_2^\alpha = \left\{ \frac{mc^2 - \mathcal{E}_1}{c}, -\mathbf{p}_1 \right\}.$$

Due the Lorentz invariance of scalar products of 4-vectors, the norm of the first 4-vector has to be equal to its value in the rest frame of particle 1, i.e. $(m_1c)^2$, i.e. that of the second vector should equal $(m_2c)^2$. This gives us a system of 2 equations for E_1 and p_1:

$$\frac{\mathcal{E}_1^2}{c^2} - p_1^2 = (m_1c)^2, \quad \frac{(mc^2 - \mathcal{E}_1)^2}{c^2} - p_1^2 = (m_2c)^2 .$$

The elementary elimination of p_1 immediately yields the required relation.

Problem O.13. Static fields **E** and **B** are uniform but arbitrary (both in magnitude and direction). What should be the velocity of an inertial reference frame to have the vectors **E***'* and **B***'*, observed from that system, parallel? Is this solution unique?

Solution: The only special direction **n**, defined by both vectors **E** and **B**, is the one perpendicular to both these vectors:

$$\mathbf{n} \equiv \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} . \qquad (*)$$

It is natural to assume that at least one of the reference frames in which **E** $||$ **B** would move along that direction: $\mathbf{v} = \mathbf{n}v$. According to Eq. (*), for that system $\mathbf{E}_\perp = \mathbf{E}$ and $\mathbf{B}_\perp = \mathbf{B}$, so that according to Eq. (9.135) of the lecture notes, the fields measured in the moving frame are

$$\begin{aligned} E'_\parallel &= E_\parallel = 0, & B'_\parallel &= B_\parallel = 0, \\ \mathbf{E}'_\perp &= \mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), & \mathbf{B}'_\perp &= \mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E} / c^2). \end{aligned}$$

The requirement to have vectors **E***'* and **B***'* parallel may be written as **E***'* × **B***'* = 0, so that for the system velocity we get the following equation:

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \times (\mathbf{B} - \mathbf{v} \times \mathbf{E} / c^2) = 0 . \qquad (**)$$

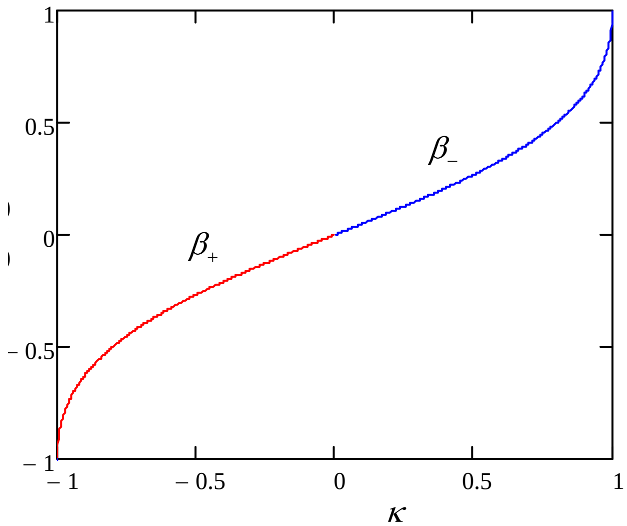
Opening the parentheses, we can notice that all 4 vectors of the result are directed along vector **n.** Also, $\mathbf{E}\times(\mathbf{v}\times\mathbf{E}) = -vE^2\mathbf{n}$, because vector $\mathbf{v}\times\mathbf{E}$ is perpendicular to **E**. Similarly, $(\mathbf{v}\times\mathbf{B})\times\mathbf{B} = vB^2\mathbf{n}$. Finally, $(\mathbf{v}\times\mathbf{B})\times(\mathbf{v}\times\mathbf{E}) = v^2\mathbf{B}\times\mathbf{E}$, because the angle between vectors $\mathbf{v}\times\mathbf{B}$ and $\mathbf{v}\times\mathbf{E}$ equals to that between vectors **B** and **E**. As a result, Eq. (**) is reduced to a quadratic equation for the reduced velocity $\beta \equiv v/c$:

$$\beta^2 - \frac{2}{\kappa}\beta + 1 = 0, \text{ where } \boldsymbol{\kappa} \equiv \frac{2(\mathbf{E}/c)\times\mathbf{B}}{(E/c)^2 + B^2}.$$

Formally, this equation has two solutions:

$$\beta_\pm = \kappa^{-1} \pm \left(\kappa^{-2} - 1\right)^{1/2}.$$

For any values and directions of vectors **E** and **B**, parameter κ, by its definition, is restricted to segment [-1, +1], and the magnitude of one of the solutions (plotted as functions of κ in Fig. below) is below 1, and hence the required reference frame does exit.



Now we can notice that since **E'** || **B'**, according the Lorentz transform (9.135), in any other reference frame 0" moving relative to frame 0' with velocity parallel to these two vectors, the transversal fields would not appear, i.e. any such system also is a solution to our problem.

Problem O.14. Analyze motion of a relativistic particle in uniform, mutually perpendicular fields **E** and **B**, for the particular case when E/c is *exactly* equal to B.

Solution: Directing coordinate axes as in Fig. 11, i.e. axis y along vector **E**, and axis z along vector **B** (so that $E_y = E$ and $B_z = B = E/c$ are the only nonvanishing field components), we get $\mathbf{u}\times\mathbf{B} = \mathbf{n}_x u_y E/c - \mathbf{n}_y u_x E/c$, so that Eq. (9.144) of the lecture notes, decomposed into 3 scalar equations, becomes

$$\frac{dp_x}{dt} = q\frac{u_y}{c}E, \quad \frac{dp_y}{dt} = qE\left(1 - \frac{u_x}{c}\right), \quad \frac{dp_z}{dt} = 0, \qquad (*)$$

while Eq. (9.148) yields the following equation for the kinetic energy $\mathcal{E}$:

$$\frac{d\mathcal{E}}{dt} = qEu_y. \qquad (**)$$

Combining the first of Eqs. (*) with Eq. (**), we get

$$\mathcal{E} - cp_x = \text{const} \equiv a, \qquad (***)$$

while the last of Eqs. (*) yields p_z = const, so that we may form another (positive) constant of motion, $(mc^2)^2 + c^2 p_z^2 \equiv b \geq 0$. Plugging these two constants into Eq. (9.78), rewritten as

$$\mathcal{E}^2 - c^2 p_x^2 \equiv (\mathcal{E} - cp_x)(\mathcal{E} + cp_x) = (mc^2)^2 + c^2 p_y^2 + c^2 p_z^2,$$

we get

$$a(\mathcal{E} + cp_x) = c^2 p_y^2 + b. \qquad (****)$$

Now considering Eqs. (***) and (****) as a system of two linear equations for $\mathcal{E}$ and p_x, we may express these variables via p_y (plus the constants of motion):

$$\mathcal{E} = \frac{c^2 p_y^2 + b}{2a} + \frac{a}{2}, \quad cp_x = \frac{c^2 p_y^2 + b}{2a} - \frac{a}{2}.$$

The first of these relations may be plugged into the product of $\mathcal{E}$ by derivative dp_y/dt, expressed from the differential equation (*) for p_y, and then transformed using the universal relativistic relation $\boldsymbol{p} = (\mathcal{E}/c^2)\mathbf{u}$ which was mentioned in Sec. 9.3,

$$\mathcal{E}\frac{dp_y}{dt} = \mathcal{E} qE\left(1 - \frac{u_x}{c}\right) = qE\left(\mathcal{E} - \mathcal{E}\frac{u_x}{c}\right) = qE(\mathcal{E} - cp_x) = qEa. \qquad (*****)$$

The result of this substitution,

$$\left(\frac{c^2 p_y^2 + b}{2a} + \frac{a}{2}\right)\frac{dp_y}{dt} \equiv \frac{d}{dt}\left[\frac{c^2}{6a} p_y^3 + \left(\frac{b}{2a} + \frac{a}{2}\right)p_y\right] = qEa.$$

may be readily integrated over time to obtain time dependence of p_y in an implicit form:

$$\frac{c^2}{6a} p_y^3 + \left(\frac{b}{2a} + \frac{a}{2}\right)p_y = qEa(t - t_0),$$

where t_0 is the time moment when $p_y = 0$. Even without solving this cubic equation, we may use the relation $dt/\mathcal{E} = dp_y/qEa$ (which follows from Eq. (*****) above) to express particle's coordinates as functions of p_y:

$$x = \int u_x dt = \int \frac{c^2 p_x}{\mathcal{E}} dt = \frac{c}{qEa}\int cp_x dp_y = \frac{c}{qEa}\int\left(\frac{c^2 p_y^2 + b}{2a} - \frac{a}{2}\right)dp_y = \frac{c}{qEa}\left[\frac{c^2}{6a} p_y^3 + \left(\frac{b}{2a} - \frac{a}{2}\right)p_y\right] + \text{const},$$

$$y = \int u_y dt = \int \frac{c^2 p_y}{\mathcal{E}} dt = \frac{c}{qEa}\int cp_y dp_y = \frac{c^2}{2qEa} p_y^2 + \text{const},$$

$$z = \int u_z dt = \int \frac{c^2 p_z}{\mathcal{E}} dt = \frac{c}{qEa}\int cp_z dp_y = \frac{c^2 p_z}{qEa} p_y + \text{const}.$$

Since z is a linear function of p_y, the first two of these relations yield particle's trajectory in the form $x(z)$, $y(z)$. Note that at large times, the magnitude of p_y (and hence of z) grows at $(t - t_0)^{1/3}$, y as $(t -$

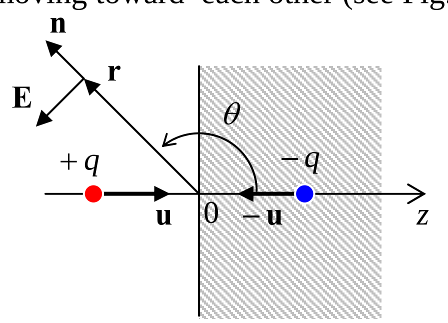
$t_0)^{2/3}$, while motion along axis x (i.e. in the direction perpendicular to both **E** and **B** vectors) becomes the fastest:

$$x \to \frac{c^3}{6qEa^2}\, p_y^3 \to c(t - t_0)\,.$$

This is essentially the drift which was discussed in Sec. 9.6(iii), whose velocity formally obeys Eq. (9.168) and (9.174), but since in the reference frame moving with this velocity ($v = c$) both the electric and magnetic fields disappear, frequency (9.172) vanishes, so that the particle does not perform the simultaneous cyclotron motion.

Problem O.15. An electron, launched directly toward a plane surface of a perfect conductor is instantly absorbed by it at the collision. Find the angular distribution and frequency spectrum of EM waves radiated at this collision, if the initial kinetic energy T of the particle is much larger than conductor's workfunction ϕ. Give a semi-quantitative discussion of the limitations of your result.

Solution: As has been discussed in class in the context of transition radiation, at not very high frequencies ($\omega << \omega_p$), the EM field outside of the metal can be viewed as a result of a head-on collision of the particle with charge q with its mirror image, with charge (-q), moving toward each other (see Fig. on the right).



Due to condition ϕ << T, the change of the particle velocity, due to its Coulomb interaction with its image, may be neglected. Hence for the initial velocities of the particle and its image, which we need for using Eq. (10.72), we can write

$$\boldsymbol{\beta}_{\text{charge}} = \frac{u}{c}\mathbf{n}_z\,, \quad \boldsymbol{\beta}_{\text{image}} = -\frac{u}{c}\mathbf{n}_z\,.$$

At collision both the particle's charge, and that of its image, disappear – in our approximation, instantly.[6] Hence the corresponding "finite" terms in Eq. (10.72) (which are proportional to charge) disappear.

Since the motion of these two charges is correlated, we have to sum up their electric fields rather than radiated powers - in Eq. (10.72), inside the modulus sign, taking into account the difference of their charge signs:

$$I(\omega) = \frac{1}{4\pi^2 c}\frac{1}{4\pi\varepsilon_0}\left|\sum_j q_j \frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}_j)}{1-\boldsymbol{\beta}_j\cdot\mathbf{n}}\right|^2 = \frac{1}{4\pi^2 c}\frac{1}{4\pi\varepsilon_0}\left|q_{\text{charge}}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}_{\text{charge}})}{1-\boldsymbol{\beta}_{\text{charge}}\cdot\mathbf{n}} + q_{\text{image}}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}_{\text{image}})}{1-\boldsymbol{\beta}_{\text{charge}}\cdot\mathbf{n}}\right|^2$$

$$= \frac{1}{4\pi^2 c}\frac{q^2\beta^2}{4\pi\varepsilon_0}\left|\frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_z)}{1-\beta\,\mathbf{n}_z\cdot\mathbf{n}} + \frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{n}_z)}{1-\beta\,\mathbf{n}_z\cdot\mathbf{n}}\right|^2 = \frac{1}{4\pi^2 c}\frac{q^2}{4\pi\varepsilon_0}\frac{\beta^2\sin^2\theta}{1-\beta^2\cos^2\theta}\,.$$

So, the spectral density of the radiation is frequency-independent, as at any short collision (corresponding to a short pulse radiated at the collision), but in our current problem this conclusion is

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[6] This approximation is only valid at time scale much larger than that of all time constants characterizing charge dynamics in conductor. (If the collision rate is negligible, this is the reciprocal plasma frequency ω_p – see Sec. 7.2.)

limited from above by frequencies ~ ω_p. Concerning the angular distribution, for nonrelativistic particles it follows the usual $\sin^2\theta$ pattern, but in the ultrarelativistic case is squeezed to the usual hollow cone with $\cos^2\theta \to 1$. Note that our results are only valid outside of the perfect conductor (into which the waves do not penetrate), so that the cone corresponds to angles $\theta \approx \pi$.

Another point deserving discussion: function $I(\omega)$ does not vanish (and in the nonrelativistic case even reaches its maximum) at $\theta = \pi/2$, i.e. in directions along conductor's surface, and one may wonder whether this is also an artifact of some approximation. Actually (again, at frequencies $\omega << \omega_p$) it is not. Indeed, as we already know, vector **E** of the electric field of the wave generated at any linear motion of the charge, is within the common plane of the observer and the line of motion – see, e.g., Eq. (10.20a). On the other hand, it is perpendicular to vector **n** of wave propagation – see Fig. above. Hence at $\theta = \pi/2$, wave's electric field is perpendicular to conductor's surface (and its magnetic field is parallel to the surface). As we know from the discussion in Sec. 7.6 with is fully consistent with the boundary conditions (7.100) on perfect conductor's surface. Nevertheless, note that this means that the propagation of the EM pulse from the collision point is accompanied by ring wave pulses of surface electric charge and current on conductor's surface.