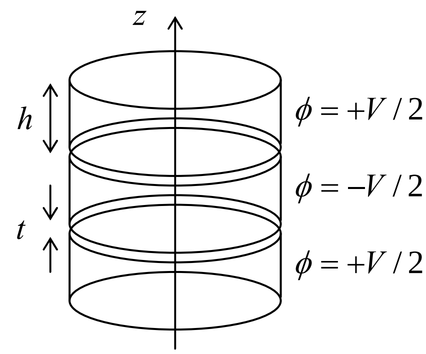
Optional problems Set 2 with solutions

Problem O.7. Solve the problem shown in Fig. 2.19 of the lecture notes (reproduced on the right); in particular, find the distribution of the electrostatic potential at the cylinder's axis.



Solution: Due to the problem's $2h$-periodicity along axis z, its particular solutions should be proportional to linear combinations of $\sin kz$ and $\cos kz$ with $2kh = 2\pi m$, where m are integers. Moreover, if the origin of axis z is selected exactly at the gap between the rings, the solution should be an odd function of z, so that for the internal problem ($\rho \le R$, where R is the rings' radius) we may write

$$\phi(\rho, z) = \sum_{m=1}^{\infty} c_m I_0\left(\frac{\pi m \rho}{h}\right) \sin\frac{\pi m z}{h}.$$

The modified Bessel function of the first kind, used in this expression, has zero index ν due to the evident axial symmetry of the problem (so that in Eq. (2.128) of the lecture notes we have to take $\mathcal{F}$ = const). The argument of this function follows from the discreteness of the variable separation constant k: $\xi = k\rho = \pi m\rho/2h$. Finally, we had to drop the modified Bessel functions of the second kind, $K_0(\xi)$, from our solution, because they diverge at $\rho \to 0$ – see Fig. 2.20 and thus cannot be used to represent the finite potential inside the rings.[1]

Coefficients c_m should be found from the boundary condition on the conducting rings; since the proper periodicity is already incorporated in our solution, it is sufficient to require that

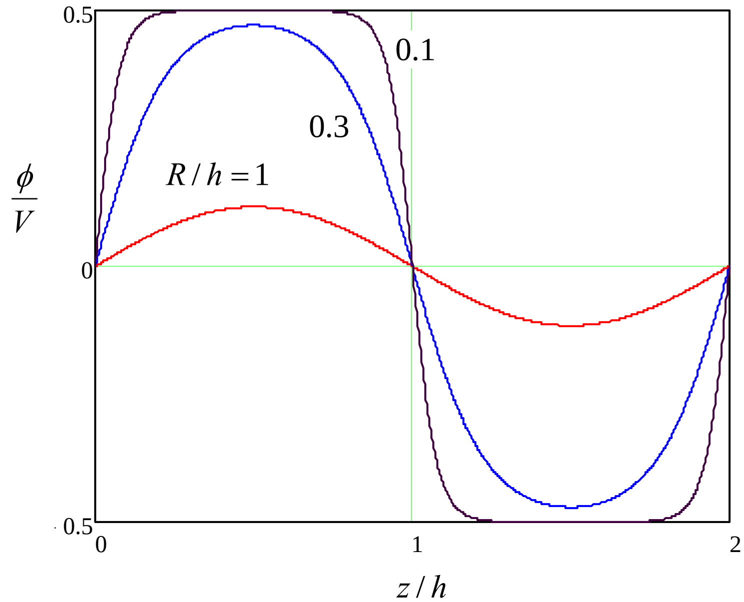
$$\phi(R, z) \equiv \sum_{m=1}^{\infty} c_m I_0\left(\frac{\pi m R}{h}\right) \sin\frac{\pi m z}{h} = +\frac{V}{2}, \quad \text{for } 0 \le z \le h.$$

Multiplying both sides of this expression by $\sin(\pi m' z/2h)$, and integrating the result from 0 to h, we get

$$c_m = \frac{V}{\pi}\left[I_0\left(\frac{\pi m R}{h}\right)\right]^{-1} \times \begin{cases} 1/m, & \text{for } m \text{ odd}, \\ 0, & \text{for } m \text{ even}. \end{cases}$$

The plot below shows the resulting electrostatic potential at the axis of the system, $\phi(0,z)$, for several values of ratio R/h. At $R > h$, the result is dominated by the first term of the series (with $m = 1$), which is proportional to $1/I_0(\pi R/2h)$, so that at $R >> h$, according to Eq. (2.158), the potential variation amplitude equals $2\pi V(R/2h)^{1/2}\exp\{-\pi R/2h\}$. On the contrary, at $R << h$, the potential follows that of the closest ring electrode at most of the axis, besides narrow ($\Delta z \sim R$) intervals centered to the gap between the electrodes.

[1] Note that these function have to be used in the solution of the *external* problem ($\rho \ge R$), which is absolutely similar to that discussed above, besides the replacement of $I_0(\pi m\rho/2h)$ with $K_0(\pi m\rho/2h)$.



Problem O.8. A fixed dipole is placed in the center of a spherical cavity of radius R carved inside a uniform, linear dielectric. Find the electric field distribution in the system (both for $r < R$ and $r > R$).

Hint: Start with the assumption that the field at $r > R$ has a distribution typical for a dipole (but be ready for surprises :-).

Solution: Following the hint, let us look for the electric potential distribution in the form

$$\phi = \frac{1}{4\pi\varepsilon_0} \times \begin{cases} \dfrac{p'\cos\theta}{r^2}, & \text{for } r \geq R, \\ \dfrac{p\cos\theta}{r^2} + \varsigma(r,\theta), & \text{for } r \leq R, \end{cases}$$

where $\varsigma(r,\theta)$ is an axially-symmetric function which should satisfy the Laplace equation. With this assumption, the boundary conditions at $r = R$ take the form

$$\phi = \text{const}: \quad \frac{p'\cos\theta}{R^2} = \frac{p\cos\theta}{R^2} + \varsigma(R,\theta),$$

$$\varepsilon\frac{\partial\phi}{\partial n} = \text{const}: \quad -2\varepsilon_r\frac{p'\cos\theta}{R^3} = -2\frac{p\cos\theta}{R^3} + \frac{\partial\varsigma}{\partial r}(R,\theta).$$

Expanding function $\varsigma(r,\theta)$ into the usual Legendre polynomial series (see, e.g., Eq. (2.172) of the lecture notes), we see from the boundary conditions that only the first term of that expansion, $a_1 r\cos\theta$, may have a nonvanishing magnitude. For a_1, the above boundary conditions yield:

$$\frac{p'}{R^2} = \frac{p}{R^2} + a_1 R, \qquad -2\varepsilon_r\frac{p'}{R^3} = -2\frac{p}{R^3} + a_1,$$

giving

$$p' = \frac{3p}{2\varepsilon_r + 1}, \quad a_1 = -\frac{2(\varepsilon_r - 1)p}{(2\varepsilon_r + 1)R^3}.$$

Hence the electric field outside the sphere indeed corresponds to a single dipole **p**' = 3**p**/(2ε_r + 1), while that inside the sphere is the sum of the original dipole field and an additional uniform field E_i = 2(ε_r - 1)/(2ε_r + 1)R^3. Notice that in the most interesting limit $\varepsilon_r \to \infty$, field E_i approaches an ε_r-independent value $E_i = p/R^3$ which gives the solution of the similar problem for a spherical cavity inside a *conductor*.

Problem O.9. A long, round cylinder is made of a ferroelectric material with fixed, constant polarization **P** perpendicular to cylinder's axis. Calculate the distribution of electric field both inside and outside the cylinder.

*Solution***:** Since there are no stand-alone charges in this problem, the electrostatic potential ϕ should satisfy the Laplace equation. Due to the evident translation symmetry of the problem in the direction of cylinder axis z, the general solution of the equation may be presented in the form of Eq. (2.112) of the lecture notes. Setting the arbitrary constant a_0 to zero, taking into account that the net charge of the cylinder (which defines constant b_0) is zero as well, and selecting the origin of angle φ in the direction of polarization vector **P**, we may reduce this solution to

$$\phi = \sum_{n=1}^{\infty} \cos n\varphi \times \begin{cases} \left(a_n \rho^n + b_n / \rho^n\right), & \text{for } \rho \le R, \\ \left(a'_n \rho^n + b'_n / \rho^n\right), & \text{for } \rho \ge R. \end{cases}$$

where R is cylinder's radius. Next, we may take into account that since the field is produced by the cylinder itself, it should vanish at large distances from the cylinder, so that all coefficients a'_n must equal zero.[2] Similarly, the potential inside the cylinder has to be finite, so that all coefficients b_n have to vanish as well. As a result, the above expression is reduced to

$$\phi = \sum_{n=1}^{\infty} \cos n\varphi \times \begin{cases} a_n \rho^n, & \text{for } \rho \le R, \\ b'_n / \rho^n, & \text{for } \rho \ge R. \end{cases}$$

In order to find coefficients a_n and b'_n, we have to write boundary conditions at the cylinder's surface ($\rho = R$). As has been discussed in lecture notes, Eq. (3.38) for the tangential component of the field is equivalent to the potential continuity, giving

$$\sum_{n=1}^{\infty} a_n R^n \cos n\varphi = \sum_{n=1}^{\infty} \frac{b'_n}{R^n} \cos n\varphi. \qquad (*)$$

However, for the normal component of the field, instead of Eq. (3.37) we have to use the more general relation D_n = const, because the cylinder material is not a linear dielectric (**D** $\neq \varepsilon$**E**). Using, for $\rho \le R$, Eq. (3.28), we get $D_n = \varepsilon_0 E_n + P_n = -\varepsilon_0 \partial\varphi/\partial\rho + P_n$. For our fixed, uniform polarization, $P_n = P\cos\varphi$, so that the second boundary condition takes the form

$$-\varepsilon_0 \sum_{n=1}^{\infty} n a_n R^{n-1} \cos n\varphi + P\cos\varphi = \varepsilon_0 \sum_{n=1}^{\infty} n \frac{b'_n}{R^{n+1}} \cos n\varphi. \qquad (**)$$

[2] Notice that coefficient a_1, which participated so prominently in many of our problems as the (minus) external field, is now equal zero as well.

Due to the mutual orthogonality of functions $\cos n\varphi$, Eqs. (*) and (**) fall apart into an infinite set of independent couples of linear equations for each pair $\{a_n, b'_n\}$, but only for $n = 1$ such a pair of equations is inhomogeneous, i.e. gives nonvanishing coefficients a_1, b'_1. (All other coefficients *may* equal zero, and due to the uniqueness of the Laplace equation solution, they *have to* equal zero.) Solving the system of equations for $n = 1$, we get:

$$a_1 = -\frac{P}{2\varepsilon_0}, \qquad b'_1 = -\frac{PR^2}{2\varepsilon_0}.$$

Finally, the electrostatic potential is

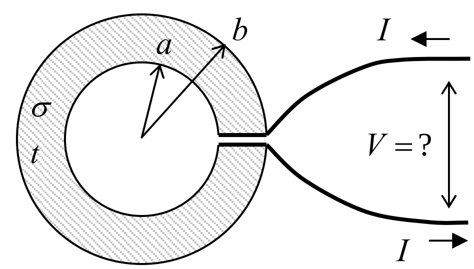
$$\phi\Big|_{\rho\le R} = -\frac{P}{2\varepsilon_0}\rho\cos\varphi, \quad \phi\Big|_{\rho\ge R} = -\frac{PR^2}{2\varepsilon_0\rho}\cos\varphi.$$

The first of these expressions describes a uniform electric field $E = P/2\varepsilon_0$, directed along the material's polarization vector $\mathbf{P}$, while the second formula gives the electric field decreasing as $1/\rho^2$. Integrating Eq. (3.7) of the lecture notes, it is straightforward to show that this is just the field of a uniform straight line (along the cylinder's axis) of parallel electric dipoles, with linear density

$$\frac{\mathbf{p}}{l} = \pi R^2 \mathbf{P}$$

– the result which could be guessed even without the formal solution of the problem.

Problem O.10. Calculate resistance of a thin, uniform, conducting foil of thickness t, cut into the shape of a ring, with a narrow cut used to connect it to two perfectly conducting electrodes (see Fig. on the right), for an arbitrary ratio b/a.



Solution: In the polar coordinates $\{\rho, \varphi\}$, the system symmetry allows the electrostatic potential ϕ inside the foil to be a function of angle φ only, because this ensures both the necessary zero values of the derivative $\partial\phi/\partial n$ at the free boundaries of the foil ($\rho = a$ and $\rho = b$), and the potential constancy at its interface with electrodes ($\varphi = 0$, $a < \rho < b$). According to MA Eq. (10.3), in this case the Laplace equation is just $d^2\phi/d\varphi^2 = 0$, with a linear solution which may be presented in the form

$$\phi = V\frac{\varphi}{2\pi} + \text{const}.$$

From here, the current density

$$j = \sigma\frac{\partial\phi}{\partial(\rho\varphi)}\Big|_{\rho=\text{const}} = \frac{\sigma V}{2\pi\rho},$$

and the total current

$$I \equiv \int_A j d^2 r = \frac{\sigma V}{2\pi} t \int_a^b \frac{d\rho}{\rho} = \frac{\sigma V}{2\pi} t \ln\frac{b}{a},$$

so that resistance V/I equals

$$\frac{2\pi}{\sigma t \ln(b/a)}.$$

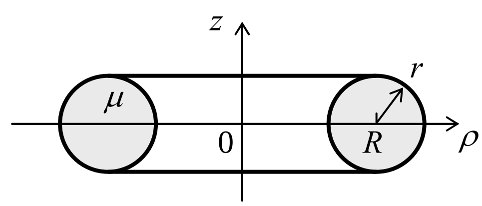
Notice that the resistance does not depend on the scale of lateral dimensions a and b. (This is a general property of Ohmic conductivity in 2D systems.) It is also curious that (for arbitrary ratio b/a), ring's Ohmic conductance I/V may be presented as a difference

$$\frac{\sigma t}{2\pi}\ln b - \frac{\sigma t}{2\pi}\ln a$$

of two conductances, one dependent only of b, and another, only on a.

In the thin-ring limit $a \to b$, the logarithm may be approximated as $\ln(b/a) \to (b-a)/a$, and the resistance is reduced to the familiar form $L/\sigma A$, where $L = 2\pi a$ is the ring's length, and $A = (b - a)t$ is the area of its cross-section.

Problem O.11. Calculate (self-)inductance of a toroidal solenoid with the cross-section shown in Fig. on the right ($r \sim R$), filled with a material of magnetic permeability μ, with $N >> 1$ wire turns uniformly distributed along the perimeter. Check your results by analyzing the limit $r << R$.



Hint : You may like to use the following table integral MA (6.11):

$$\int_0^1 \ln\frac{a+\sqrt{1-\xi^2}}{a-\sqrt{1-\xi^2}}\,d\xi = \pi\left(a-\sqrt{a^2-1}\right), \quad \text{for } a \geq 1.$$

Solution: In class, we have calculated magnetic field inside such a solenoid without magnetic filling – see Eq. (5.41) Since all the field is concentrated inside the solenoid, if it is filled with a magnetic material, we can just multiply this result by ratio μ/μ_0:

$$B = \frac{\mu NI}{2\pi\rho}.$$

We can now calculate the magnetic flux piercing one wire loop:

$$\Phi_1 = \int_A B_n d^2r = \frac{\mu NI}{\pi}\int_0^r dz \int_{R-\sqrt{r^2-z^2}}^{R+\sqrt{r^2-z^2}} \frac{d\rho}{\rho} = \frac{\mu NI}{\pi}\int_0^r \ln\frac{R+\sqrt{r^2-z^2}}{R-\sqrt{r^2-z^2}}\,dz$$

$$= \frac{\mu NIr}{\pi}\int_0^1 \ln\frac{R/r+\sqrt{1-\xi^2}}{R/r-\sqrt{1-\xi^2}}\,d\xi = \mu NI\left(R-\sqrt{R^2-r^2}\right).$$

Just as for the long solenoid discussed in class, the flux Φ piercing the whole wire is N times larger. As a result, the solenoid inductance

$$L = \frac{\Phi}{I} = \mu N^2\left(R-\sqrt{R^2-r^2}\right).$$

In the limit $r << R$, we may expand this expression into the Taylor series in small r/R, and in the first approximation get

$$L \approx \mu N^2 \frac{r^2}{2R} = \mu n^2 lA, \quad \text{with } n \equiv \frac{N}{l}, \quad l \equiv 2\pi R, \quad \text{and } A \equiv \pi r^2.$$

We see that in this limit, the result coincides with the inductance of a long straight solenoid, calculated in class – see Eq. (5.82). It is interesting that in the opposite limit ($r = R$), the result also acquires a very simple form,

$$L = \mu N^2 r .$$

Notice that these results may be used for relatively small values of N only if $\mu >> \mu_0$, otherwise we have to account for "stray" magnetic fields spilling out of the torus interior between separated wire turns.

Problem 0.12. A planar, uniform superconducting film of thickness $t \sim \delta_L$ carries a finite supercurrent J per unit width. Use the London equation to find the current distribution over the film thickness.

Solution: This problem has an evident symmetry relative to the plane passing through the middle of film's thickness. This is why, selecting this plane for $x = 0$ (where axis x is perpendicular to the film), we need to take only the even solution $\mathbf{A} = A(x)\mathbf{n}_y$ of Eq. (6.44) of the lecture notes. For $j \propto A(x)$, such solution is

$$j = \text{const} \times \cosh\frac{x}{\delta_L}.$$

The constant may be found from the condition that integral of j over the film's thickness is equal to current's linear density J. As a result, we get

$$j = \frac{J}{2\delta_L}\frac{\cosh(x/\delta_L)}{\sinh(t/2\delta_L)}.$$

This result shows that if the film is thin ($t << \delta_L$), the supercurrent is uniformly distributed over its thickness $j \approx J/t$.[3] In the opposite limit, the current is concentrated in two δ_L-layers at the opposite surfaces of the film.

[3] This result does not specify how is the supercurrent distributed over the film *width*. This is a more difficult problem, and its solution depends on the relation between δ_L^2 and the cross-section area $A = wt$ of a film strip. Only if the strip is very narrow, $w << \delta_L^2/t$, the current is uniformly distributed along the width as well. (Optional problem: try to prove this, at least on a semi-quantitative level.)