Selected Mathematical Formulas

which are used in this lecture course series, but not always remembered by students (and instructors :-)

1. Constants

- Euclidean circle's length-to-diameter ratio

$$\pi = 3.141\,592\,653...; \quad \sqrt{\pi} \approx 1.77\,. \tag{1.1}$$

- Natural logarithm base:

$$e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718\,281\,828...\ ; \tag{1.2a}$$

from that number, the logarithm base conversion factors are as follows:

$$\frac{\ln x}{\log_{10} x} = \ln 10 \approx 2.303, \quad \frac{\log_{10} x}{\ln x} = \frac{1}{\ln 10} \approx 0.434\,. \tag{1.2b}$$

- The Euler (or "Euler-Mascheroni") constant (for the definition, see EM Sec. 2.4) :

$$\gamma = 0.57721\,566490...; \quad e^{\gamma} \approx 1.781\,. \tag{1.3}$$

2. Combinatorics, sums, and series

(i) <u>Combinatorics</u>

- The number of different *permutations*, i.e. *ordered* sequences of k elements selected from a set of n distinct elements, is

$$^{n}P_k \equiv n\cdot(n-1)\cdot...\cdot(n-k+1) = \frac{n!}{(n-k)!}\,; \tag{2.1a}$$

in particular, the number of different permutations of *all* elements of the set ($n = k$) is

$$^{k}P_k = k\cdot(k-1)\cdot...\cdot 2\cdot 1 = k!\ . \tag{2.1b}$$

- The number of different *combinations*, i.e. *unordered* sequences of k elements from a set of $n \geq k$ distinct elements, is equal to the "binomial coefficient"

$$^{n}C_k \equiv \binom{n}{k} \equiv \frac{^{n}P_k}{^{k}P_k} = \frac{n!}{k!(n-k)!}, \tag{2.2}$$

In an alternative, very popular "ball/box language", $^{n}C_k$ is the number of different ways to put in a box, in an arbitrary order, k balls from the total number of n, while considering each ball distinct from others.

- A generalization of the binomial coefficient notion is the "multinomial coefficient",

$$ {}^{n}C_{k_1,k_2,\ldots k_l} \equiv \frac{n!}{k_1!k_2!\ldots k_l!}, \quad \text{with } n = k_1 + k_2 + \ldots + k_l, \tag{2.3}$$

which, in the standard mathematical language, is a number of different permutations in a multiset of l distinct element types from an n-element set which contains k_j (j = 1, 2,…l) elements of each type. In the "ball/box language", coefficient (2.3) is the number of different ways to distribute n balls between l different boxes, each time keeping the number (k_j) of balls in the j-th box fixed, but ignoring their order in the box. The binomial coefficient ${}^{n}C_k$, defined by Eq. (2.2), is evidently a particular case of the multinomial coefficient for $l = 2$, so that if $k_1 \equiv k$, then $k_2 = n - k$.

(ii) Sums and series

- Arithmetic progression:

$$ r + 2r + \ldots + nr \equiv \sum_{k=1}^{n} kr = \frac{n(r+nr)}{2}; \tag{2.4a}$$

in particular, at $r = 1$ it is reduced to the sum of n first natural numbers:

$$ 1 + 2 + \ldots + n \equiv \sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{2.4b}$$

- Sum of squares of n first natural numbers:

$$ 1^2 + 2^2 + \ldots + n^2 \equiv \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2.5}$$

- The *Riemann zeta function*:

$$ \zeta(s) \equiv 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^s}; \tag{2.6a}$$

the particular values frequently met in applications are

$$ \zeta\left(\frac{3}{2}\right) \approx 2.612, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta\left(\frac{5}{2}\right) \approx 1.341, \quad \zeta(3) \approx 1.202, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(5) \approx 1.037. \tag{2.6b}$$

- Finite geometric progression (for real $\lambda \neq 1$):

$$ 1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1} \equiv \sum_{k=0}^{n-1} \lambda^k = \frac{1-\lambda^n}{1-\lambda}; \tag{2.7a}$$

in particular, if $\lambda^2 < 1$, the progression has a finite limit at $n \to \infty$ (the "geometric series"):

$$ \lim_{n\to\infty} \sum_{k=0}^{n-1} \lambda^k = \sum_{k=0}^{\infty} \lambda^k = \frac{1}{1-\lambda}. \tag{2.7b}$$

- Binomial sum:

$$\left(1+a\right)^n = \sum_{k=0}^{n} {}^nC_k a^k \,, \tag{2.8}$$

where ${}^nC_k$ are the binomial coefficients defined by Eq. (2.2).

- The Stirling formula:

$$\lim_{n\to\infty} \ln(n!) = n(\ln n - 1) + \frac{1}{2}\ln(2\pi n) + \frac{1}{12n} - \frac{1}{360n^3} + \ldots ; \tag{2.9}$$

for most applications in physics, the first term (first derived by A. de Moivre) is sufficient.

- The Taylor (or "Taylor-Maclaurin") series: for any infinitely differentiable function $f(x)$:

$$\lim_{\tilde{x}\to 0} f(x+\tilde{x}) = f(x) + \frac{df}{dx}(x)\,\tilde{x} + \frac{1}{2!}\frac{d^2 f}{dx^2}(x)\tilde{x}^2 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k f}{dx^k}(x)\,\tilde{x}^k \,; \tag{2.10a}$$

note that for many functions this series converges only within a limited, sometimes small range of deviations $\tilde{x}$. For a function of several arguments, $f(x_1,x_2,\ldots,x_N)$, the first terms of the Taylor series are

$$\lim_{\tilde{x}_k\to 0} f(x_1+\tilde{x}_1, x_2+\tilde{x}_2,\ldots) = f\Big|_{\tilde{x}_k=0} + \sum_{k=1}^{N} \frac{\partial f}{\partial x_k}\Big|_{\tilde{x}_k=0} \tilde{x}_k + \frac{1}{2!}\sum_{k,k'=1}^{N} \frac{\partial^2 f}{\partial x_k \partial x_{k'}}\Big|_{\tilde{x}_k,\tilde{x}_{k'}=0} \tilde{x}_k \tilde{x}_{k'} + \ldots \tag{2.10b}$$

- The Euler-Maclaurin formula, valid for any infinitely differentiable function $f(x)$:

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x)dx - \frac{1}{2} f(0) - \frac{1}{6}\cdot\frac{1}{2!}\frac{df}{dx}(0) + \frac{1}{30}\cdot\frac{1}{4!}\frac{d^3 f}{dx^3}(0) + \ldots ; \tag{2.11}$$

the coefficients participating in this formula are the so-called *Bernoulli numbers*:[1]

$$B_1 = \frac{1}{2},\quad B_2 = \frac{1}{6},\quad B_3 = 0,\quad B_4 = \frac{1}{30},\quad B_5 = 0,\quad B_6 = \frac{1}{42},\quad B_7 = 0,\quad B_8 = \frac{1}{30},\ldots . \tag{2.12}$$

3. Trigonometric functions

- Sums of two functions of arbitrary arguments:

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{B-A}{2}\,, \tag{3.1a}$$

$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}\,, \tag{3.1b}$$

$$\sin A \pm \sin B = 2\sin\frac{A\pm B}{2}\cos\frac{\pm B - A}{2}\,. \tag{3.1c}$$

- Products:

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)\,, \tag{3.2a}$$

[1] Please note that definitions of B_k (or rather their signs and indices) differ even in the most popular handbooks. (This nomenclature controversy does not affect Eq. (2.11) - of course.)

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)\,, \tag{3.2b}$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)\,; \tag{3.2c}$$

for the particular case of equal arguments, $B = A$, these formulas yield expressions for squares of trigonometric functions, and their product:

$$\cos^2 A = \frac{1}{2}(\cos 2A + 1), \quad \sin^2 A = \frac{1}{2}(1 - \cos 2A), \quad \sin A\cos A = \frac{1}{2}\sin 2A. \tag{3.2d}$$

- Cubes of trigonometric functions:

$$\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A, \quad \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A. \tag{3.3}$$

4. General differentiation

- Full differential of a product of two functions:

$$d(fg) = (df)g + f(dg). \tag{4.1}$$

- Full differential of a function of several independent arguments, $f(x_1, x_2,\ldots,x_n)$:

$$df = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} dx_k\,. \tag{4.2}$$

5. General integration

- Integration by parts (immediately follows from Eq. (4.1)):

$$\int_{g(A)}^{g(B)} f\, dg = fg\Big|_A^B - \int_{f(A)}^{f(B)} g\, df\,. \tag{5.1}$$

- Numerical (approximate) integration of 1D functions: the simplest "*trapezoidal rule*",

$$\int_a^b f(x)dx \approx h\left[f\left(a+\frac{h}{2}\right) + f\left(a+\frac{3h}{2}\right) + \ldots + f\left(b-\frac{h}{2}\right)\right] = h\sum_{n=1}^{N} f\left(a - \frac{h}{2} + nh\right), \quad h \equiv \frac{b-a}{N}. \tag{5.2}$$

has relatively low accuracy (error of the order of $(h^3/12)d^2f/dx^2$ per step), so that the following "*Simpson formula*",

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f(a+h) + 2f(a+2h) + \ldots + 4f(b-h) + f(b)\right], \quad h \equiv \frac{b-a}{2N}, \tag{5.3}$$

whose error per step scales as $(h^5/180)d^4f/dx^4$, is used much more frequently.[2]

[2] Higher-order formulas (e.g., the "Bode rule"), and other guidance including ready-for-use codes for computer calculations may be found, for example, in the popular reference texts by W. H. Press *et al*., cited in the References section below. Besides that, some advanced codes are used as subroutines in the software packages

6. A few 1D integrals of elementary functions[3]

(i) Indefinite integrals

- Integrals with $(1 + \xi^2)^{1/2}$:

$$\int \left(1+\xi^2\right)^{1/2} d\xi = \frac{\xi}{2}\left(1+\xi^2\right)^{1/2} + \frac{1}{2}\ln\left|\xi + \left(1+\xi^2\right)^{1/2}\right|, \tag{6.1}$$

$$\int \frac{d\xi}{\left(1+\xi^2\right)^{1/2}} = \ln\left|\xi + \left(1+\xi^2\right)^{1/2}\right|. \tag{6.2}$$

- Integrals with $(\xi^2 + 2a\xi - 1)^{1/2}$:

$$\int \frac{d\xi}{\xi\left(\xi^2 + 2a\xi - 1\right)^{1/2}} = \arccos\frac{a\xi - 1}{|\xi|\left(a^2+1\right)^{1/2}}. \tag{6.3}$$

(ii) Semi-definite integrals:

- Integrals with $1/(e^{\xi} \pm 1)$:

$$\int_a^{\infty} \frac{d\xi}{e^{\xi}+1} = \ln\left(1+e^{-a}\right), \tag{6.4a}$$

$$\int_{a>0}^{\infty} \frac{d\xi}{e^{\xi}-1} = \ln\frac{1}{1-e^{-a}}. \tag{6.4b}$$

(iii) Definite integrals

- Integrals with $1/(1 + \xi^2)$:

$$\int_0^{\infty} \frac{d\xi}{1+\xi^2} = \frac{\pi}{2}. \tag{6.5a}$$

$$\int_0^{\infty} \frac{d\xi}{\left(1+\xi^2\right)^{3/2}} = 1. \tag{6.5b}$$

- Integrals with $(1 - \xi^{2n})^{1/2}$:

$$\int_0^{1} \frac{d\xi}{[1-\xi^{2n}]^{1/2}} = \frac{\sqrt{\pi}}{2n}\Gamma\left(\frac{1}{2n}\right)/\Gamma\left(\frac{n+1}{2n}\right). \tag{6.6}$$

where $\Gamma(s)$ is the *gamma-function* (which is frequently defined by Eq. (6.7)) whose its main property is

listed in the same section. In some cases, the Euler-Maclaurin formula (2.11) also may be useful for numerical integration.

[3] A powerful (and free :-) interactive online tool for working out indefinite 1D integrals is available at http://integrals.wolfram.com/index.jsp.

$$\Gamma(n) = (n-1)!, \quad \text{for } n = 1, 2, \ldots; \tag{6.6b}$$

other particularly important values of the gamma-function are

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{1\cdot 3}{2\cdot 2}\sqrt{\pi}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\sqrt{\pi}, \ \ldots\ . \tag{6.6c}$$

- Integrals with $e^{-\xi}$:

$$\int_0^\infty \xi^{s-1} e^{-\xi} d\xi = \Gamma(s), \tag{6.7}$$

- Integrals with $1/(e^{\xi} \pm 1)$:

$$\int_0^\infty \frac{\xi^{s-1} d\xi}{e^{\xi}+1} = \left(1-2^{1-s}\right)\Gamma(s)\varsigma(s), \quad \text{for } s > 0, \tag{6.8a}$$

$$\int_0^\infty \frac{\xi^{s-1} d\xi}{e^{\xi}-1} = \Gamma(s)\varsigma(s), \quad \text{for } s > 1, \tag{6.8b}$$

where $\zeta(s)$ is the Riemann zeta-function – see Eq. (2.6). Particular cases: for $s = 2n$,

$$\int_0^\infty \frac{\xi^{2n-1} d\xi}{e^{\xi}+1} = \frac{2^{2n-1}-1}{2n}\pi^{2n} B_{2n}, \tag{6.8c}$$

$$\int_0^\infty \frac{\xi^{2n-1} d\xi}{e^{\xi}-1} = \frac{(2\pi)^{2n}}{4n} B_{2n}. \tag{6.8d}$$

where B_n are the Bernoulli numbers – see Eq. (2.12). For the particular case $s = 1$ (when Eq. (6.8a) yields uncertainty),

$$\int_0^\infty \frac{d\xi}{e^{\xi}+1} = \ln 2. \tag{6.8e}$$

- Integrals with $\exp\{-\xi^2\}$:

$$\int_0^\infty \xi^s e^{-\xi^2} d\xi = \frac{1}{2}\Gamma\left(\frac{s+1}{2}\right), \quad \text{for } s > -1; \tag{6.9a}$$

the most important particular cases are $s = 0$,

$$\int_0^\infty e^{-\xi^2} d\xi = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}, \tag{6.9b}$$

and $s = 2$:

$$\int_0^\infty \xi^2 e^{-\xi^2} d\xi = \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{4}. \tag{6.9c}$$

- Integrals with $\cos\xi$:

$$\int_0^\infty \frac{\cos\xi}{a^2+\xi^2}d\xi = \frac{\pi}{2a}e^{-a}. \quad (6.10)$$

7. 3D vector products

(i) Definitions:

- Scalar product:

$$\mathbf{a}\cdot\mathbf{b} = \sum_{j=1}^{3} a_j b_j \ , \quad (7.1)$$

where a_j and b_j are vector components in any orthogonal coordinate system. In particular, vector squared (the same as norm squared):

$$a^2 \equiv \mathbf{a}\cdot\mathbf{a} = \sum_{j=1}^{3} a_j^2 \equiv \|\mathbf{a}\|^2. \quad (7.2)$$

- Vector ("cross-") product:

$$\mathbf{a}\times\mathbf{b} \equiv \mathbf{n}_1(a_2b_3 - a_3b_2) + \mathbf{n}_2(a_3b_1 - a_1b_3) + \mathbf{n}_3(a_1b_2 - a_2b_1) = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad (7.3)$$

where $\{\mathbf{n}_j\}$ is the set of mutually perpendicular unit vectors[4] along the corresponding coordinate system directions.[5] In particular, Eq. (7.3) yields

$$\mathbf{a}\times\mathbf{a} = 0. \quad (7.4)$$

(ii) Corollaries (readily verified by Cartesian components):

- Double vector product (the so-called "bac minus cab rule"):

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \quad (7.5)$$

- Mixed scalar-vector product (the "operator rotation rule"):

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}). \quad (7.6)$$

- Scalar product of vector products:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}); \quad (7.7a)$$

in the particular case of two similar operands (say, $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$), the last formula is reduced to

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[4] Popular alternative notations for this vector set are $\{\mathbf{e}_j\}$ and $\{\hat{\mathbf{r}}_j\}$.

[5] It is easy to use Eq. (7.3) to check that the direction of the product vector corresponds to the "corkscrew rule": if we rotate the first operand toward the second one, the usual corkscrew moves in the direction of the product.

$$\left(\mathbf{a}\times\mathbf{b}\right)^2 = (ab)^2 - (\mathbf{a}\cdot\mathbf{b})^2 . \quad (7.7b)$$

8. Differentiation in 3D Cartesian coordinates

- Definition of the "del" (or "nabla") vector-operator: [6]

$$\nabla \equiv \sum_{j=1}^{3} \mathbf{n}_j \frac{\partial}{\partial r_j}, \quad (8.1)$$

where r_j is the linear ("Cartesian") coordinate along direction $\mathbf{n}_j$. In accordance with this definition, del acting on a *scalar* function of coordinates, $f(\mathbf{r})$,[7] gives its gradient:

$$\nabla f \equiv \sum_{j=1}^{3} \mathbf{n}_j \frac{\partial f}{\partial r_j} \equiv \mathbf{grad}\, f , \quad (8.2)$$

i.e., a new *vector*.

- The "scalar product" of del by a *vector* function

$$\mathbf{f}(\mathbf{r}) \equiv \sum_{j=1}^{3} \mathbf{n}_j f_j(\mathbf{r}), \quad (8.3)$$

compiled formally following Eq. (7.1), is a *scalar* function – the "divergence" of the initial function:

$$\nabla\cdot\mathbf{f} \equiv \sum_{j=1}^{3} \frac{\partial f_j}{\partial r_j} \equiv \mathbf{div}\,\mathbf{f} , \quad (8.4)$$

while the "vector product" of ∇ and **f**, formed in a formal accordance with Eq. (7.3), is a new vector - the curl[8] of **f**:

$$\nabla\times\mathbf{f} \equiv \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ \dfrac{\partial}{\partial r_1} & \dfrac{\partial}{\partial r_2} & \dfrac{\partial}{\partial r_3} \\ f_1 & f_2 & f_3 \end{vmatrix} = \mathbf{n}_1\left(\frac{\partial f_3}{\partial r_2} - \frac{\partial f_2}{\partial r_3}\right) + \mathbf{n}_2\left(\frac{\partial f_1}{\partial r_3} - \frac{\partial f_3}{\partial r_1}\right) + \mathbf{n}_3\left(\frac{\partial f_2}{\partial r_1} - \frac{\partial f_1}{\partial r_2}\right) \equiv \mathbf{curl}\,\mathbf{f}. \quad (8.5)$$

- One more frequently met[9] "product" is $(\mathbf{f}\cdot\nabla)\mathbf{g}$, where **f** and **g** are two arbitrary vector functions of **r**. This product should be also understood in the sense dictated by Eq. (7.1), i.e. as a vector whose *j*-th component is

$$\left[(\mathbf{f}\cdot\nabla)\mathbf{g}\right]_j = \sum_{j'=1}^{3} f_{j'} \frac{\partial g_j}{\partial r_{j'}}. \quad (8.5)$$

[6] One can also meet the following notation: $\nabla \equiv \partial/\partial\mathbf{r}$, which may be convenient is some cases, but misleading in others, so it will not be used in these lecture notes.
[7] In this, and next 4 sections, all scalar and vector functions are assumed to be differentiable.
[8] In the European tradition, this operator is called "rotor" and denoted as **rot**.
[9] See, e.g., Eqs. (11.5) and (11.6) below.

9. The Laplace operator

- Definition in Cartesian coordinates (in a formal accordance with Eq. (7.2)):

$$\nabla^2 \equiv \nabla \cdot \nabla = \sum_{j=1}^{3} \frac{\partial^2}{\partial r_j^2} . \qquad (9.1)$$

- According to the definition, the Laplace operator acting on a *scalar* function of coordinates gives a new scalar function:

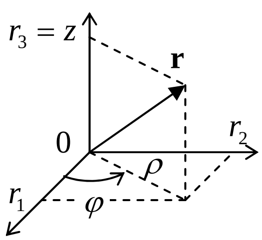
$$\nabla^2 f \equiv \nabla \cdot (\nabla f) = \mathbf{div}\,(\mathbf{grad}\ f) = \sum_{j=1}^{3} \frac{\partial^2 f}{\partial r_j^2} . \qquad (9.2)$$

- On the other hand, acting on a *vector* function (8.3), operator ∇^2 returns another vector:

$$\nabla^2 \mathbf{f} = \sum_{j=1}^{3} \mathbf{n}_j \nabla^2 f_j \ . \qquad (9.3)$$

10. Operators ∇ and ∇^2 in the most important systems of orthogonal coordinates[10]

(i) Cylindrical[11] coordinates $\{\rho, \varphi, z\}$ (see Fig. on the left) may be defined by their relations with the Cartesian coordinates:



$$\begin{aligned} r_1 &= \rho \cos\varphi, \\ r_2 &= \rho \sin\varphi, \\ r_3 &= z. \end{aligned} \qquad (10.1)$$

- Gradient of a scalar function:

$$\nabla f = \mathbf{n}_\rho \frac{\partial f}{\partial \rho} + \mathbf{n}_\varphi \frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \mathbf{n}_z \frac{\partial f}{\partial z}. \qquad (10.2)$$

- The Laplace operator of a scalar function:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}, \qquad (10.3)$$

- Divergence of a vector function:

$$\nabla \cdot \mathbf{f} = \frac{1}{\rho}\frac{\partial (\rho f_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial f_\varphi}{\partial \varphi} + \frac{\partial f_z}{\partial z}. \qquad (10.4)$$

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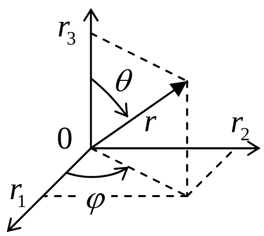
[10] Some other orthogonal coordinate systems are discussed in EM Sec. 2.3.
[11] In 2D geometry with fixed coordinate *z*, these coordinates are called “polar”.

- Curl of a vector function:

$$\nabla \times \mathbf{f} = \mathbf{n}_\rho \left(\frac{1}{\rho}\frac{\partial f_z}{\partial \varphi} - \frac{\partial f_\varphi}{\partial z} \right)\frac{\partial f}{\partial \rho} + \mathbf{n}_\varphi \left(\frac{\partial f_\rho}{\partial z} - \frac{\partial f_z}{\partial \rho} \right) + \mathbf{n}_z \frac{1}{\rho}\left(\frac{\partial(\rho f_\varphi)}{\partial \rho} - \frac{\partial f_\rho}{\partial \varphi} \right). \qquad (10.5)$$

(ii) <u>Spherical coordinates</u> $\{r, \theta, \varphi\}$ (see Fig. on the left) may be defined as:



$$r_1 = r\sin\theta\cos\varphi,$$
$$r_2 = r\sin\theta\sin\varphi, \qquad (10.6)$$
$$r_3 = r\cos\theta.$$

- Gradient of a scalar function:

$$\nabla \mathbf{f} = \mathbf{n}_r \frac{\partial f}{\partial r} + \mathbf{n}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{n}_\varphi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}. \qquad (10.7)$$

- The Laplace operator of a scalar function:[10]

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{(r\sin\theta)^2}\frac{\partial^2 f}{\partial \varphi^2}. \qquad (10.8)$$

- Divergence of a vector function:

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2}\frac{\partial\left(r^2 f_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(f_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial f_\varphi}{\partial \varphi}. \qquad (10.9)$$

- Curl of a vector function:

$$\nabla \times \mathbf{f} = \mathbf{n}_r \frac{1}{r\sin\theta}\left(\frac{\partial(f_\varphi \sin\theta)}{\partial \theta} - \frac{\partial f_\theta}{\partial \varphi} \right) + \mathbf{n}_\theta \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial f_r}{\partial \varphi} - \frac{\partial(r f_\varphi)}{\partial r} \right) + \mathbf{n}_\varphi \frac{1}{r}\left(\frac{\partial(r f_\theta)}{\partial r} - \frac{\partial f_r}{\partial \theta} \right). \qquad (10.10)$$

11. Products involving vector ∇

(i) <u>Useful zeros</u>:

- For any scalar function $f(\mathbf{r})$,

$$\nabla \times (\nabla f) \equiv \mathbf{curl}\,(\mathbf{grad}\ f) = 0. \qquad (11.1)$$

- For any vector function $\mathbf{f}(\mathbf{r})$,

$$\nabla \cdot (\nabla \times \mathbf{f}) \equiv \mathbf{div}\,(\mathbf{curl}\,\mathbf{f}) = 0. \qquad (11.2)$$

(ii) <u>Laplace operator via the curl of curl:</u>

$$\nabla^2 \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla \times (\nabla \times \mathbf{f}). \qquad (11.3)$$

(iii) Differentiation of a product of a scalar function by a vector function:

- The scalar 3D generalization of Eq. (4.1) is

$$\nabla\cdot\left(f\,\mathbf{g}\right)=\left(\nabla f\right)\cdot\mathbf{g}+f\left(\nabla\cdot\mathbf{g}\right), \tag{11.4a}$$

and its vector generalization is similar:

$$\nabla\times\left(f\,\mathbf{g}\right)=\left(\nabla f\right)\times\mathbf{g}+f\left(\nabla\times\mathbf{g}\right). \tag{11.4b}$$

(iv) 3D differentiation of products of two vector functions:

- The most important formula of this class is

$$\nabla\times\left(\mathbf{f}\times\mathbf{g}\right)=\mathbf{f}\left(\nabla\cdot\mathbf{g}\right)-\left(\mathbf{f}\cdot\nabla\right)\mathbf{g}-\left(\nabla\cdot\mathbf{f}\right)\mathbf{g}+\left(\mathbf{g}\cdot\nabla\right)\mathbf{f}\,, \tag{11.5}$$

but the following two formulas also may be met:

$$\nabla\cdot\left(\mathbf{f}\cdot\mathbf{g}\right)=\left(\mathbf{f}\cdot\nabla\right)\mathbf{g}+\left(\mathbf{g}\cdot\nabla\right)\mathbf{f}+\mathbf{f}\times\left(\nabla\times\mathbf{g}\right)+\mathbf{g}\times\left(\nabla\times\mathbf{f}\right), \tag{11.6}$$

$$\nabla\cdot\left(\mathbf{f}\times\mathbf{g}\right)=\left(\nabla\times\mathbf{f}\right)\cdot\mathbf{g}-\mathbf{f}\cdot\left(\nabla\times\mathbf{g}\right). \tag{11.7}$$

12. Integro-differential relations

(i) For an arbitrary surface *A* limited by closed contour *C*:

- The Stokes theorem:

$$\int_A\left(\nabla\times\mathbf{f}\right)\cdot d\mathbf{A}\equiv\int_A\left(\nabla\times\mathbf{f}\right)_n d^2r=\oint_C\mathbf{f}\cdot d\mathbf{r}\equiv\oint_C f_\tau dr\,, \tag{12.1}$$

where $d\mathbf{A} = \mathbf{n}d^2r$ is the elementary area vector (normal to the surface), and $d\mathbf{r}$ is the elementary contour length vector (tangential to the contour line).

(ii) For an arbitrary volume *V* limited by closed surface *A*:

- Divergence theorem:

$$\int_V\left(\nabla\cdot\mathbf{f}\right)d^3r=\oint_A\mathbf{f}\cdot d\mathbf{A}\equiv\oint_A f_n d^2r\,. \tag{12.2}$$

- Green's theorem for two arbitrary scalar functions *f* and *g*:[12]

$$\int_V\left(f\,\nabla^2 g-g\nabla^2 f\right)d^3r=\oint_A\left(f\,\nabla g-g\nabla f\right)\cdot d^2r\,. \tag{12.3}$$

[12] This formula readily follows from the divergence theorem (12.2) by applying it to vector functions $f\nabla g$ and $g\nabla f$, then using Eq. (11.4a) in the left-hand parts of each result, and subtracting them.

13. The Kronecker delta and Levi-Civita symbols

- The Kronecker delta-symbol (defined for integer indices):

$$\delta_{jj'} \equiv \begin{cases} 1, & \text{if } j' = j, \\ 0, & \text{otherwise.} \end{cases} \tag{13.1}$$

- The Levi-Civita permutation symbol (most frequently used for 3 integerindices, each taking values 1, 2, or 3):

$$\varepsilon_{jj'j''} \equiv \begin{cases} +1, & \text{if all 3 indices are different and follow in any of "correct" orders: 123, 231, or 312,} \\ -1, & \text{if all 3 indices are different and follow in any of "incorrect" orders: 321, 213, or 132,} \\ 0, & \text{if any pair of indices coincide.} \end{cases} \tag{13.2}$$

14. Dirac's delta function

- Definition of 1D δ-function (for real $a < b$):

$$\int_a^b f(x)\delta(x)dx = \begin{cases} f(0), & \text{if } a < 0 < b, \\ 0, & \text{otherwise,} \end{cases} \tag{14.1}$$

where $f(x)$ is any function continuous near $x = 0$. In particular (if $f(x) = 1$ near $x = 0$), the definition yields

$$\int_a^b \delta(x)dx = \begin{cases} 1, & \text{if } a < 0 < b, \\ 0, & \text{otherwise.} \end{cases} \tag{14.2}$$

- A very important integral:

$$\int_{-\infty}^{+\infty} e^{i\omega t} dt = 2\pi\delta(\omega), \tag{14.3a}$$

where i is the imaginary unity. The coefficient in this equation may be readily checked (or recalled:-) by considering it the Fourier-integral presentation of $f(t) \equiv 1$, and applying Eq. (14.1) to the reciprocal Fourier transform

$$1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega t}\left[2\pi\delta(\omega)\right]d\omega. \tag{14.3b}$$

- 3D generalization of the delta-function (the 2D generalization is similar):

$$\int_V f(\mathbf{r})\delta(\mathbf{r})d^3r = \begin{cases} f(0), & \text{if } 0 \in V, \\ 0, & \text{otherwise;} \end{cases} \tag{14.4}$$

its presentation via a product of 1D delta-functions of Cartesian coordinates:

$$\delta(\mathbf{r}) = \delta(r_1)\delta(r_2)\delta(r_3). \tag{14.5}$$

15. The Cauchy integral

- For any complex function $f(w)$ which is analytical within such part of the complex plane w, that is limited by closed contour C and includes point $w = z$,

$$f(z) = \frac{1}{2\pi i}\oint_C f(w)\frac{dw}{w-z}. \tag{15.1}$$

16. References

(i) For more formulas, and their discussions, I can recommend the following handbooks (in the alphabetic order):

- M. Abramowitz and I. S. Stegun (eds.), *Handbook of Mathematical Formulas*, Dover, 1965 (and numerous later printings);[13]
- I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 5th ed., Academic Press, 1980;
- A. Jeffrey and H. H. Dai, *Handbook of Mathematical Formulas and Integrals*, 4th ed., Academic Press, 2008;
- G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, 2nd ed., Dover, 2000.

Many formulas are also available from the symbolic calculation parts of commercially available software packages listed below.

On a personal note, perhaps 90% of all formula needs during my 45+ year research career have been satisfied by a small, wonderfully compiled book:

- H. B. Dwight, *Tables of Integrals and Other Mathematical Formulas*, 4th ed., MacMillan, 1961, whose used copies, rather surprisingly, are still available on the Web.

(ii) Properties of some special functions are briefly discussed in the most relevant points of the lecture notes:

- Bessel functions: EM Sec. 2.4;
- Hermite polynomials: QM Sec. 2.6;
- Laguerre polynomials (both simple and associated): QM Sec. 3.5;
- Legendre polynomials, associated Legendre functions, and spherical harmonics: EM Sec. 2.4 and QM Sec. 3.5.

(iii) Perhaps the most popular code collections for numerical calculations are the twin manuals

- W. H. Press *et al.*, *Numerical Recipes in FORTRAN*, 2nd ed., Cambridge U. Press, 1992;
- W. H. Press *et al.*, *Numerical Recipes* [in C++ - KKL], 3rd ed., Cambridge U. Press, 2007.

[13] An updated version of this collection is now available online at http://dlmf.nist.gov/ .

My lecture notes include only very brief introductions into numerical methods of solutions of differential equations:

- ordinary differential equations: CM Sec. 3.9;
- equations with partial derivatives: CM Sec. 3.4 and EM Sec. 2.7.

(iv) The most popular software packages for numerical and symbolic calculations (in the alphabetic order):

- *Maple* (the current official Web site: http://www.maplesoft.com/);
- *Mathcad* (http://www.ptc.com/products/mathcad/);
- *Mathematica* (http://www.wolfram.com/products/mathematica/index.html);
- *MATLAB* (http://www.mathworks.com/products/matlab/).